Goldstein classical mechanics solutions manual pdf

Goldstein Classical Mechanics NotesMichael Good May 30, 2004 1 1.1 Chapter 1: Elementary Principles Mechanics of a Single Particle Classical mechanics incorporates special relativity. 'Classical' refers to the contradistinction to 'quantum' mechanics. Velocity: v= Linear momentum: p = mv. dp . dt In most cases, mass is constant and force is simplified: F= F= Acceleration: d2 r . dt2 Newton's second law of motion holds in a reference frame that is inertial or Galilean. a= Angular Momentum: L = r × p. Torque: T = r × F. Torque is the time derivative of angular momentum: d dv (mv) = m = ma. dt dt Force: dr . dt 1 T= Work: dL . dt 2 W12 = 1 F · dr. In most cases, mass is constant and work simplifies to: W12 = m 2 dv dv · vdt = m v· dt = m dt dt 1 1 m 2 2 W12 = (v2 − v1) = T2 − T1 2 2 2 v · dv 1 Kinetic Energy: mv 2 2 The work is the change in kinetic energy. T = A force is considered conservative if the work is the same for any physically possible path. Independence of W12 on the particular path implies that the work done around a closed ciruit is zero: F · dr = 0 If friction is present, a system is non-conservative. Potential Energy: F = − V (r). The capacity to do work that a body or system has by viture of is position is called its potential energy. V above is the potential energy. To express work in a way that is independent of the path taken, a change in a quantity that depends on only the end points is needed. This quantity is potential energy. Work is now V1 − V2 . The change is -V. Energy Conservation Theorem for a Particle: If forces acting on a particle are conservative, then the total energy of the particle, T + V, is conserved. The Conservation Theorem for the Linear Momentum of a Particle states that linear momentum, p, is conserved if the total force F, is zero. The Conservation Theorem for the Angular Momentum of a Particle states that angular momentum, L, is conserved if the total torque T, is zero. 2 1.2 Mechanics of Many Particles Newton's third law of motion, equal and opposite forces, does not hold for all forces. It is called the weak law of action and reaction. Center of mass: R= mi ri = mi mi ri . M Center of mass moves as if the total external force were acting on the entire mass of the system concentrated at the center of mass. Internal forces that obey Newton's third law, have no effect on the motion of the center of mass. F(e) ≡ M d2 R = dt2 Fi . i (e) Motion of center of mass is unaffected. This is how rockets work in space. Total linear momentum: P= i mi dri dR =M . dt dt Conservation Theorem for the Linear Momentum of a System of Particles: If the total external force is zero, the total linear momentum is conserved. The strong law of action and reaction is the condition that the internal forces between two particles, in addition to being equal and opposite, also lie along the line joining the particles. Then the time derivative of angular momentum is the total external torque: dL = N(e) . dt Torque is also called the moment of the external force about the given point. Conservation Theorem for Total Angular Momentum: L is constant in time if the applied torque is zero. Linear Momentum Conservation requires weak law of action and reaction. Angular Momentum Conservation requires strong law of action and reaction. Total Angular Momentum: L= i ri × pi = R × M v + i ri × pi . 3 plus the K. of motion about the center of mass. has two parts: the K. think particle placed on surface of a sphere because it will eventually slide down part of the way but will fall off. plus the angular momentum of motion about the center of mass. .Total angular momentum about a point O is the angular momentum of motion concentrated at the center of mass. 1. • nonholonomic constraints: think walls of a gas container.. The term on the right is called the internal potential energy. not moving along the curve of the sphere. i. like angular momentum. 1.. rheonomous constraints: time is an explicit variable. Total Work: W12 = T2 − T1 where T is the total kinetic energy of the system: T = Total kinetic energy: T = 1 2 2 mi vi = i 1 2 i 2 mi vi . 1 1 M v2 + 2 2 mi vi2 .. r2 . If the center of mass is at rest wrt the origin then the angular momentum is independent of the point of reference. think a particle constrained to move along any curve or on a given surface.j i=j If the external and internal forces are both derivable from potentials it is possible to define a total potential energy such that the total energy T + V is conserved. For rigid bodies the internal potential energy will be constant.example: bead on rigid curved wire fixed in space Difficulties with constraints: 4 .. scleronomous constraints: equations of contraint are NOT explicitly dependent on time.. For a rigid body the internal forces do no work and the internal potential energy remains constant.E. Total potential energy: V = i Vi + 1 2 Vij . i Kinetic energy.E.3 Constraints • holonomic constraints: think rigid body. t) = 0. obtained if all the mass were concentrated at the center of mass...example: bead on moving wire 2. r3 . think f (r1 . and no friction systems. 2. For holonomic constraints introduce generalized coordinates. This is the only restriction on the nature of the constraints: workless in a virtual displacement. The result is: {[d ∂T ∂T ( )−] − Qj }δqj = 0 dt ∂ qj ˙ ∂qj 5 . Quanities with with dimensions of energy or angular momentum. 1. Two angles for a double pendulum moving in a plane. For nonholonomic constraints equations expressing the constraint cannot be used to eliminate the dependent coordinates. Generalized coordinates are worthwhile in problems even without constraints. Two angles expressing position on the sphere that a particle is constrained to move on. This is called a transformation. Once we have the expression in terms of generalized coordinates the coefficients of the δqi can be set separately equal to zero. Equations of motion are not all independent. like rigid body systems. and must be obtained from solution. 3. Degrees of freedom are reduced.1. This is again D'Alembert's principle for the motion of a system.4 D'Alembert's Principle and Lagrange's Equations dpi) · δri = 0 dt Developed by D'Alembert. This is great news. the principle that: (Fi i (a) − This is valid for systems which virtual work of the forces of constraint vanishes. Amplitudes in a Fourier expansion of rj . Examples of generalized coordinates: 1. going from one set of dependent variables to another set of independent variables. Use independent variables. and what is good about it is that the forces of constraint are not there. The generalized coordinates are independent of each other for holonomic constraints. Transform this equation into an expression involving virtual displacements of the generalized coordinates. but it is not yet in a form that is useful for deriving equations of motion. because coordinates are no longer all independent 2. Forces are not known beforehand. and thought of first by Bernoulli. eliminate dependent coordinates. Nonholonomic constraints are HARDER TO SOLVE. 4. and L contains the potential of the conservative forces as before. Friction is commonly. For a charge mvoing in an electric and magnetic field. and notice they are identical component wise: m¨ = q[Ex + (v × B)x]. L=T −U where U is the generalized potential or velocity-dependent potential. If you remember the individual coefficients vanish. dt ∂ qj ˙ ∂qj where Qj represents the forces not arising from a potential. the Lorentz force dictates: F = q[E + (v × B)]. the electromagnetic field. Ff x = −kx vx . Rayleigh's dissipation function: Fdis = 1 2 2 2 2 (kx vix + ky viy + kz viz). x If frictional forces are present(not all the forces acting on the system are derivable from a potential). and forgive me for skipping some steps. The equation of motion can be dervied for the x-dirction.Lagrange's Equations come from this principle. i The total frictional force is: Ff = − Work done by system against friction: dWf = −2Fdis dt 6 v Fdis .5 Velocity-Dependent Potentials and The Dissipation Function The velocity dependent potential is important for the electromagnetic forces on moving charges. the result is: ∂L d ∂L ( )− =0 dt ∂ qj ˙ ∂qj 1. Lagrange's equations can always be written: d ∂L ∂L ( )− = Qj . and allow the forces derivable from a scaler potential function. 7 . do not appear in the Lagrangian formulation. ∂ qj ˙ In use. a single particle is space(Cartesian coordinates. atwood's machine 3. Form L from them. Simple examples are: 1. They also cannot be directly derived. 3.The rate of energy dissipation due to friction is 2Fdis and the component of the generalized force resulting from the force of friction is: Qj = − ∂Fdis . Forces of contstraint.6 Applications of the Lagrangian Formulation The Lagrangian method allows us to eliminate the forces of constraint from the equations of motion. Scalar functions T and V are much easier to deal with instead of vector forces and accelerations. 2. Procedure: 1. Solve for the equations of motion. Write T and V in generalized coordinates. Plane polar coordinates) 2. a bead sliding on a rotating wire(time-dependent constraint). Put L into Lagrange's Equations 4. dt ∂ qj ˙ ∂qj ∂ qj ˙ 1. both L and Fdis must be specified to obtain the equations of motion: d ∂L ∂L ∂Fdis ( )− =− . Show that for a single particle with constant mass the equation of motion implies the follwing differential equation for the kinetic energy: dT =F·v dt while if the mass varies with time the corresponding equation is d(mT) = F · p. 2004 1 Derivations 1. dt dt 2 2.j Answer: MR = mi ri 1 .Goldstein Chapter 1 Derivations Michael Good June 27. d(mT) d p2 ˙ = ( ) = p · p = F · p. dt Answer: d(1 mv 2) dT ˙ = 2 = mv · v = ma · v = F · v dt dt with time variable mass. i. Prove that the magnitude R of the position vector for the center of mass from an arbitrary origin is given by the equation: M 2 R2 = M i 2 mi ri − 1 2 2 mi mj rij . Answer: First.j mi mj ri · rj Solving for ri · rj realize that rij = ri − rj .j 3. The strong law demands they be equal and opposite and lie along the line joining the particles.M 2 R2 = i. The argument may be generalized to a system with arbitrary number of particles.j 1 M 2 i 1 2 M 2 R2 = M i 2 mi ri − 1 2 2 mi mj rij i. The equations governing the individual particles are ˙ p1 = F1 + F21 ˙ p2 = F2 + F12 (e) (e) 2 . The first equation of motion tells us that internal forces have no effect. Suppose a system of two particles is known to obey the equations of motions. The weak law demands that only the forces be equal and opposite. thus proving the converse of the arguments leading to the equations above.j 2 mi mj rij i. Square ri − rj and you get 2 2 2 rij = ri − 2ri · rj + rj Plug in for ri · rj 1 2 2 2 (r + rj − rij) 2 i 1 1 2 2 mi mj ri + mi mj rj − 2 i.j 2 ri · rj = 1 2 mi r i + M 2 2 mj rj − j M 2 R2 = M 2 R2 = 1 2 2 mi mj rij i. d2 R (e) Fi ≡ F(e) M 2 = dt i dL = N(e) dt From the equations of the motion of the individual particles show that the internal forces between particles satisfy both the weak and the strong laws of action and reaction.j i. if the particles satisfy the strong law of action and reaction then they will automatically satisfy the weak law. The equations of constraint for the rolling disk. so that the angle between them is zero. xn)dxi = 0. For two particles. A × B = ABsinθ (e) (e) 4. . i=1 A constraint condition of this type is holonomic only if an integrating function f (x1 . that is. . Assuming the equation of motion to be true. xn) can be found that turns it into an exact differential. ie the magnitude of their cross product is zero. . Clearly the function must be such that ∂(f gi) ∂(f gj) = ∂xj ∂xi for all i = j. . the internal torque contribution is r1 × F21 + r2 × F12 = r1 × F21 + r2 × (−F21) = (r1 − r2) × F21 = r12 × F21 = 0 Now the only way for r12 × F21 to equal zero is for both r12 and F21 to lie on the line joining the two particles. . . Answer: 3 . then ˙ ˙ p1 + p2 = F1 + F21 + F2 + F12 must give F12 + F21 = 0 Thus F12 = −F21 and they are equal and opposite and satisfy the weak law of action and reaction. Show that no such integrating factor can be found for either of the equations of constraint for the rolling disk. the internal torque contribution is null. dx − a sin θdψ = 0 dy + a cos θdψ = 0 are special cases of general linear differential equations of constraint of the form n gi (x1 . . If the particles obey dL = N(e) dt then the time rate of change of the total angular momentum is only equal to the total external torque. Performing the same procedure on the second equation you can find ∂f ∂(f a cos θ) = ∂y ∂φ a cos θ and f sin θ = 0 4 ∂f ∂f = ∂y ∂φ . φ) The only way for f to satisfy this equation is if f is constant and thus apparently there is no integrating function to make these equations exact. Q is −a sin θ and W is 0. The equations that are equivalent to ∂(f gi) ∂(f gj) = ∂xj ∂xi are ∂(f P) ∂(f Q) = ∂φ ∂x ∂(f P) ∂(f W) = ∂θ ∂x ∂(f Q) ∂(f W) = ∂θ ∂φ These are explicitly: ∂(f) ∂(−f a sin θ) = ∂φ ∂x ∂(f) =0 ∂θ ∂(−f a sin θ) =0 ∂θ Simplfying the last two equations yields: f cos θ = 0 Since y is not even in this first equation.First attempt to find the integrating factor for the first equation. the integrating factor does not depend on y and because of ∂f = 0 it does not depend on θ either. Thus ∂θ f = f (x. Note it is in the form: P dx + Qdφ + W dθ = 0 where P is 1. and φ have meanings similar to those in the problem of a single vertical disk. φ and φ . cos θdx + sin θdy = 0 1 a(dφ + dφ) 2 (where θ. (x ± b b cos θ. The whole combination rolls without slipping on a palne.y) are the corrdinates of a point on the axle midway between the two wheels) and one holonomic equation of constraint.φ. one for each disk ˙ v = aφ ˙ v = aφ and two contact points. Two wheels of radius a are mounted on the ends of a common axle of length b such that the wheels rotate independently. 5. and solve for the equations of motion. Show that there are two nonholonomic equations of constraint. φ) and making it impossible for f to satsify the equations unless as a constant. sin θdx − cos θdy = a θ = C − (φ − φ) b where C is a constant. find the point of contact. That makes me feel better. Once you have the equations of motion. it was confusing to me too. y ± sin θ) 2 2 5 . and y component of position. Answer: The trick to this problem is carefully looking at the angles and getting the signs right. I think the fastest way to solve this is to follow the same procedure that was used for the single disk in the book. Here the steps are taken a bit further because a holonomic relationship can be found that relates θ. and (x.∂f =0 ∂θ leading to f = f (y. find the speed of the disk. from there its just slightly tricky algebra. Here goes: We have two speeds. and take the derivative of the x component. that is. If this question was confusing to you. Mary Boas says it is 'not usually worth while to spend much time searching for an integrating factor' anyways. So just think about it. I also have the primed wheel south-west of the non-primed wheel. Make sure you get the angles right.The contact points come from the length of the axis being b as well as x and y being the center of the axis. Do it for the next one and get: y+ ˙ b ˙ ˙ cos θθ = −aφ cos θ 2 All of the derivatives together so you aren't confused what I just did: y− ˙ x− ˙ x+ ˙ b ˙ ˙ sin θθ = aφ sin θ 2 b ˙ ˙ sin θθ = aφ sin θ 2 b ˙ ˙ y + cos θθ = −aφ cos θ ˙ 2 b ˙ ˙ y − cos θθ = −aφ cos θ ˙ 2 Now simplify them by cancelling the dt s and leaving the x and y's on one side: 6 . This will give us the components of the velocity. A picture would help. and the points of contact. and get: x+ ˙ b ˙ ˙ sin θθ = aφ sin θ 2 The plus sign is there because of the derivative of cos multiplied with the negative for the primed wheel distance from the center of the axis. b d (x + cos θ) = vx dt 2 x− ˙ b ˙ sin θθ = v cos(180 − θ − 90) = v cos(90 − θ) = v cos(−90 + θ) = v sin θ 2 b ˙ ˙ x − sin θθ = aφ sin θ ˙ 2 Do this for the next one. but I can't do that on latex yet. The components of the distance are cos and sin for x and y repectively. For the y parts: d b (y + sin θ) = vy dt 2 b ˙ ˙ cos θθ = −v cos θ 2 It is negative because I decided to have axis in the first quadrent heading south-east. So now that we've found the speeds. they were tricky for me. we want to take the derivatives of the x and y parts of their contact positions. Show that for f (t) differentiable.b dx = sin θ[dθ + adφ] (1) 2 b (2) dx = sin θ[− dθ + adφ] 2 b dy = − cos θ[dθ + adφ] (3) 2 b dy = − cos θ[− dθ + adφ] (4) 2 Now we are done with the physics. 7 . but otherwise arbitrary. A particle moves in the xy plane under the constraint that its velocity vector is always directed towards a point on the x axis whose abscissa is some given function of time f (t). For the holonomic equation use (1)-(2). (1) − (2) = 0 = bdθ + a(dφ − dφ) a dθ = − (dφ − dφ) b a θ = − (φ − φ) + C b For the other two equations. The rest is manipulation of these equations of motion to come up with the constraints. I started with b b (1) cos θ + (3) sin θ = cos θ sin θ[dθ + adφ] − sin θ cos θ[dθ + adφ] 2 2 cos θdx + sin θdy = 0 and (1) + (2) = 2dx = sin θa[dφ + dφ] (3) + (4) = 2dy = − cos θa[dφ + dφ] multiply dy by − cos θ and multiply dx by sin θ to yield yourself a − cos θdy = cos2 θ [dφ + dφ] 2 2 a sin θdx = sin θ [dφ + dφ] 2 Add them together and presto! sin θdx − cos θdy = a [dφ + dφ] 2 6. It has the distance f (t). Thus the constraint is nonholonomic. 8 . ˙ ∂T ∂T −2 =Q ∂q ˙ ∂q Show this. The Lagrangian equations can be written in the form of the Nielsen's equations. Answer: The abscissa is the x-axis distance from the origin to the point on the x-axis that the velocity vector is aimed at. then vy Vy = vx Vx y(t) dy = dx x(t) − f (t) dy dx = y(t) x(t) − f (t) This cannot be integrated with f (t) being arbituary. I claim that the ratio of the velocity vector components must be equal to the ratio of the vector components of the vector that connects the particle to the point on the x-axis. 7. It's nice to write the constraint in this way because it's frequently the type of setup Goldstein has: ydx + (f (t) − x)dy = 0 There can be no integrating factor for this equation. The velocity vector components are: dy dt dx vx = dt The vector components of the vector that connects the particle to the point on the x-axis are: vy = Vy = y(t) Vx = x(t) − f (t) For these to be the same.the constraint is nonholonomic. The directions are the same. t). q. t) = ˙ + q+ ˙ q ¨ dt ∂t ∂q ∂q ˙ Now lets solve for ˙ ∂T ∂q ˙ . so lets plug this into equation (5). d ∂T ∂ ∂T ∂ ∂T ∂T ( )= ( )+ ( )q + ˙ ( )¨ q dt ∂ q ˙ ∂t ∂ q ˙ ∂q ∂ q ˙ ∂q ˙ ∂q ˙ because T = T (q. q. d ∂T ∂T ∂T ˙ T ≡ T (q. d ˙ T ≡ T (q. you must not forget the chain rule. not forgetting the product rule ˙ ∂T ∂ ∂T ∂T ∂T = [+ q+ ˙ q] ¨ ∂q ˙ ∂ q ∂t ∂q ∂q ˙ ∂T ∂ ∂T ∂ ∂T ∂T ∂ ∂T = + q+ ˙ + q ¨ ∂q ˙ ∂q ∂t ˙ ∂ q ∂q ˙ ∂q ∂q ∂ q ∂q ∂q ˙ ˙ ˙ ∂T ∂ ∂T ∂ ∂T ∂ ∂T = + q+ ˙ + ( )¨ q ∂q ˙ ∂q ∂t ˙ ∂q ∂q ˙ ∂q ∂q ˙ ∂q ˙ Now we have ˙ ∂T ∂q ˙ . 9 . That will show that they can be written as displayed above. d ∂T ∂T ∂ ∂T ∂ ∂T ∂ ∂T ( )+ = + q+ ˙ + ( )¨ q dt ∂ q ˙ ∂q ∂t ∂ q ˙ ∂q ∂ q ˙ ∂q ∂q ˙ ∂q ˙ ∂q ∂q ˙ ˙ d ∂T ∂ ∂T ∂ ∂T ( )= + q+ ˙ ( )¨ q dt ∂ q ˙ ∂t ∂ q ˙ ∂q ∂ q ˙ ∂q ˙ ∂q ˙ Notice that this is indeed true.Answer: I'm going to set the two forms equal and see if they match. lets solve for T first. Lagrangian Form = Nielsen's Form ˙ d ∂T ∂T ∂T ∂T ( )− = −2 dt ∂ q ˙ ∂q ∂q ˙ ∂q ˙ ∂T ∂T d ∂T ( )+ = dt ∂ q ˙ ∂q ∂q ˙ What is ˙ ∂T ∂q ˙ (5) ˙ you may ask? Well. q. t) dt Because d dt is a full derivative. . Answer: Let's directly substitute L into Lagrange's equations. function of its arguments.. d dF d dF − =0 dt dq dt dq Is the same as ˙ ∂F ∂F = ∂q ˙ ∂q This is shown to be true because ˙ ∂F ∂F = ∂q ˙ ∂q We have ˙ d ∂F d ∂F = dt ∂ q ˙ dt ∂q = ∂ ∂F ∂ ∂F + q ˙ ∂t ∂q ∂q ∂q ∂q = ˙ ∂ ∂F ∂F ∂F [+ q] = ˙ ∂q ∂t ∂q ∂q ∂q 10 .. ∂L d ∂L − =0 dt ∂ q ˙ ∂q d ∂ dF ∂ dF (L +)− (L +)=0 dt ∂ q ˙ dt ∂q dt d ∂L d ∂ dF ∂L d ∂ dF [+]− − − =0 dt ∂ q ˙ ∂ q dt ∂q dt d ∂ dF ∂ dF d ∂L ∂L − + − =0 dt ∂ q ˙ ∂q dt ∂ q dt ∂ q ˙ dt On the left we recognized Lagrange's equations. qn .If L is a Lagrangian for a system of n degrees of freedom satisfying Lagrange's equations. which we know equal zero. Now to show the terms with F vanish. t) dt also satisfies Lagrange's equations where F is any arbitrary. but differentiable. show by direct substitution that L =L+ dF (q1 . . . t)] ˙ v 1 q q ∂ψ q mv 2 − qφ + A · v + + ψ(r. 9.. qn . This is all that you need to show that the Lagrangian is changed but the motion is not. This problem is now in the same form as before: dF (q1 . . t) · v] q ˙ L = L + [ψ] c In the previous problem it was shown that: ˙ ˙ d ∂ψ ∂ψ = dt ∂ q ˙ ∂q For ψ differentiable but arbitrary. L = T − V is a suitable Lagrangian. . but it is not the only Lagrangian for a given system. The electromagnetic field is invariant under a gauge transformation of the scalar and vector potential given by A→A+ ψ→φ− ψ(r.. L =L+ 10. there is no unique Lagrangian). t) dt And if you understood the previous problem. Let q1 .Thus as Goldstein reminded us..e. you'll know why there is no effect on the motion of the particle(i. What effect does this gauge transformation have on the Lagrangian of a particle moving in the electromagnetic field? Is the motion affected? Answer: q 1 mv 2 − qφ + A · v 2 c Upon the gauge transformation: L= L = L = 1 ∂ψ q 1 mv 2 − q[φ −] + [A + 2 c ∂t c ψ(r. qn be a set of independent generalized coordinates for a system 11 .. t) 1 ∂ψ c ∂t where ψ is arbitrary (but differentiable). t) · v 2 c c ∂t c q ∂ψ + L =L+ [c ∂t ψ(r... there are many Lagrangians that may describe the motion of a system. .of n degrees of freedom.. . then L satisfies Lagrange's equations with respect to the s coordinates d ∂L ∂L − =0 dt ∂ sj ˙ ∂sj In other words.. Answer: We know: ∂L d ∂L − =0 dt ∂ qi ˙ ∂qi and we want to prove: d ∂L ∂L − =0 dt ∂ sj ˙ ∂sj ∂L ∂L If we put ∂ sj and ∂sj in terms of the q coordinates. i = 1. (Such a transformatin is called a point transformation. t).. n. q.) Show that if the Lagrangian function is expressed as a function of sj . ∂L = ∂sj ∂L = ∂ sj ˙ We know: i ∂L ∂qi ∂qi ∂sj ∂L ∂ qi ˙ ∂ qi ∂ sj ˙ ˙ i ∂ qi ˙ ∂qi = ∂ sj ˙ ∂sj Thus. with a Lagrangian L(q.. sj and t through the equa˙ tion of transformation. the form of Lagrange's equations is invariant under a point transformation. . t). ∂L = ∂ sj ˙ Plug ∂L ∂ sj ˙ i ∂L ∂qi ∂ qi ∂sj ˙ and ∂L ∂sj into the Lagrangian equation and see if they satisfy it: d [dt ∂L ∂qi]−[∂ qi ∂sj ˙ 12 ∂L ∂qi]=0 ∂qi ∂sj i i .. sn .. then they can be ˙ substituted back in and shown to still satisfy Lagrange's equations. . Suppose we transform ˙ to another set of independent coordinates s1 . sn by means of transformation equations qi = qi (s1 . Pulling out the summation to the right and [i ∂qi ∂sj to the left. 13 . we are left with: d ∂L ∂L ∂qi −] =0 dt ∂ qi ˙ ∂qi ∂sj This shows that Lagrangian's equations are invariant under a point transformation. • Derive Lagrange's equations and find the generalized force. L=T −V T = Therefore L= Plug into the Lagrange equations: d ∂L ∂L − =Q dt ∂ x ˙ ∂x d ∂ 1 mr2 ω 2 d ∂ 1 mr2 ω 2 2 − 2 =Q dt ∂(rω) ∂x d m(rω) = Q dt m(rω) = Q ¨ 1 m(rω)2 2 1 1 mv 2 = m(rω)2 2 2 V =0 1 . Consider a uniform thin disk that rolls without slipping on a horizontal plane. we need to first find the Lagrangian. Answer: To find Lagrange's equations. • Discuss the motion if the force is not applied parallel to the plane of the disk. A horizontal force is applied to the center of the disk and in a direction parallel to the plane of the disk. 2004 1 Exercises 11.Goldstein Chapter 1 Exercises Michael Good July 17. The velocity of the disk would not just be in the x-direction as it is here. that in order to reach the escape velocity the ratio of the wight of the fuel to the weight of the empty rocket must be almost 300! m Answer: This problem can be tricky if you're not very careful with the notation. Answer: 1 GM m = mv 2 r 2 GM 1 = v2 r 2 Lets plug in the numbers to this simple problem: (6. 12. The escape velocity of a particle on Earth is the minimum velocity required at Earth's surface in order that that particle can escape from Earth's gravitational field. me + mf . But here is the best way to do it. such as θ to describe the y-axis motion. neglecting atmospheric friction. From the conservation theorme for potential plus kinetic energy show that the escape veolcity for Earth. Show. Rockets are propelled by the momentum reaction of the exhaust gases expelled from the tail.118 × 104 m/s which is 11. ingnoring the presence of the Moon.2 km/s. Integrate this equation to obtain v as a function of m. is: dv dm = −v − mg dt dt where m is the mass of the rocket and v' is the velocity of the escaping gases relative to the rocket. Since these gases arise from the raction of the fuels carried in the rocket.1 km/s and a mass loss per second equal to 1/60th of the intial mass. 13. and finally the goal is to find the ratio of 2 .2 km/s. Neglecting the resistance of the atmosphere. Defining me equal to the empty rocket mass. then there might be some slipping. is 11.If the motion is not applied parallel to the plane of the disk. that is. and m0 dm dt = − 60 as the loss rate of mass. the mass of the rocket is not constant. with v' equal to 2. or another generalized coordinate would have to be introduced. assuming a constant time rate of loss of mass.67 × 10−11) · (6 × 1024) 1 = v2 (6 × 106) 2 This gives v = 1. for a rocket starting initally from rest. m0 is the intitial rocket mass. the system is conservative. mf is the total fuel mass. but decreases as the fuel is expended. Show that the equation of motion for a rocket projected vertically upward in a uniform gravitational field. as in Newton's second law. The total force on the rocket will be equal to the force due to the gas escaping minus the weight of the rocket: ma = m d [−mv] − mg dt dm dv = −v − mg dt dt The rate of lost mass is negative. me v 60g dm + dm m m0 me m0 dv = −v m0 dm + m 60g dm m0 v = −v ln v = −v ln me 60g + (me − m0) m0 m0 me − me − mf me + 60g me + mf me + mf me + mf mf − 60g me me + mf v = v ln Now watch this. so. The velocity is in the negative direction. I'm going to use my magic wand of approximation. Use this: dv dm dv = dm dt dt Solve: m dv dm dm = −v − mg dm dt dt v dm dv dm =− −g dm dt m dt v 60g dv =− + dm m m0 Notice that the two negative signs cancelled out to give us a positive far right term. I can ignore the empty 3 . dv = − Integrating.mf /me to be about 300. This is when I say that because I know that the ratio is so big. with the two negative signs the term becomes positive. The total force is just ma. 07 km/s which is a more accurate approximation. and T2 is the kinetic energy about the center of mass. Keep these two parts seperate! Solve for T1 first.1 km/s in his third edition without checking his answer. while the angle θ will be the angle from the z-axis.8 for g. T1 + T2 = T Where T1 equals the kinetic energy of the center of mass. This is more like the number 300 he was looking for. The angle φ will be the angle in the x-y plane. realizing that the rigid rod is not restricted to just the X-Y plane. its the easiest: 1 1 2 ˙ ˙ M vcm = (2m)(aψ)2 = ma2 ψ 2 2 2 Solve for T2 . the center of which is constrained to move on a circle of radius a.rocket mass as compared to the fuel mass. 9. mf /me . Don't forget the Z-axis! T1 = T2 = 1 M v 2 = mv 2 2 Solve for v 2 about the center of mass. if Goldstein hadn't just converted 6800 ft/s from his second edition to 2. and 2100 m/s for v . me α0 . the solution actually follows more quickly. but if α = α0 then only k ≈ 1. again. If we start at Goldstein's equation. This graph is arccosh(k)/k = α and looks like a little hill.This symmetric but physically equivalent example is not what the problem asked for. no real values of k exist. cosh2 A − sinh2 A = 1 a more manageable expression in terms of k and α becomes apparent. 5 . the dimensional quantities defined in the problem. It can be graphed by typing acosh(x)/x on a free applet at If α < α0 . k= we have k = cosh kα Taking the derivative with respect to k.ctc.html.edu/home/jkim/gcalc.81 Since α0 = √ 1 k2 −1 = α0 ≈ . 1 = α0 sinh kα0 Using the hyperbolic identity.tacoma. but I think its interesting. x = a cosh y a x xy = cosh a ax Using. t) have been referred to ˙ ¨ as 'jerky' mechanics. qi . x. we have δq = Applying this we have 2 ∂q dα ∂α δI = 1 (∂L ∂L ∂ q ˙ ∂L ∂ q ¨ δq + dα + dα)dt ∂q ∂ q ∂α ˙ ∂ q ∂α ¨ 6 . By applying the mehtods of the calculus of variations. 2 ∂q dt ˙ i dt ∂ qi ˙ ∂qi Apply this result to the Lagrangian k m qq − q2 ¨ 2 2 Do you recognize the equations of motion? L=− Answer: If there is a Lagrangian of the form L = L(qi . Problems for which triple dot x = f (x.12 The term generalized mechanics has come to designate a variety of classical mechanics in which the Lagrangian contains time derivatives of qi higher than the first. qi qi . In analogy with the differential quantity. x. qi . . Goldstein Equation (2. t)dt ˙ ¨ and ∂I dα = ∂α 2 (1 i ∂L ∂qi ∂L ∂ qi ˙ ∂L ∂ qi ¨ dα + dα + dα)dt ∂qi ∂α ∂ qi ∂αi ˙ ∂ qi ∂αi ¨ To make life easier. Chapter 11). t).12). qi . then we have: 2 i = 1. we're going to assume the Einstein summation convention. 2. and ˙¨ Hamilton's principle holds with the zero variation of both qi and qi at the end ˙ points. I= 1 L(qi . then the corresponding Euler-Lagrange equations are d ∂L d2 ∂L ( )− ( )+ = 0. Such equations of motion have interesting applications in chaos theory (cf. show that if there is a Lagrangian of the form L(qi .. qi . n.2. as well as drop the indexes entirely.. t) ˙ ¨ and Hamilton's principle holds with the zero variation of both qi and qi at ˙ the end points. . 2 − 1 ∂ 2 q d ∂L d ∂L ( )dt = ∂t∂α dt ∂ q ∂α ¨ 2 2 − 1 1 − ∂q d2 ∂L dt ∂α dt2 ∂ q ¨ First term vanishes for the third time. we get closer 2 δI = 1 (∂L d ∂L d2 ∂L δq − δq + 2 δq)dt ∂q dt ∂ q ˙ dt ∂ q ¨ Gathering δq's. Turn the crank again. Substituting back in. throwing our summation sign and index's back in. one by one: d ∂L d2 ∂L ∂L − + 2 =0 ∂qi dt ∂ qi ˙ dt ∂ qi ¨ 7 i = 1. Integration by parts on the middle term yields. and we are still left with another integration by parts problem. (Goldstein.The indexes are invisible and the two far terms are begging for some mathematical manipulation. Eq. . 2. Now the last term needs attention. and using the definition of our δq. Here goes: 2 1 ∂L ∂ q ¨ ∂L 2 ∂ q dt = ∂ q ∂α ¨ 2 2 − 1 1 ∂ 2 q d ∂L ( )dt ∂t∂α dt ∂ q ∂α ¨ Where we used vdu = uv − vdu as before. we have: 2 δI = 1 (d ∂L ∂L ∂ q ˙ ∂L δq − δq + 2 δq)dt ∂q dt ∂ q ˙ ∂ q ∂α ¨ ∂q Where we used the definition δq = ∂α dα again.10) and see that δI = 0 requires that the coefficients of δqi separately vanish. 2. and applying Hamilton's principle: 2 δI = 1 i (d ∂L d2 ∂L − + 2)δqi dt = 0 ∂qi dt ∂ qi ˙ dt ∂ qi ¨ We know that since qi variables are independent. and we have 2 1 ¨ ∂L ∂ q dt = ∂ q ∂α ¨ 2 1 ∂q d2 ∂L ∂α dt2 ∂ q ¨ Plugging back in finally. . The first term vanishes once again. in analogy to Goldstein page 44.n. 2 1 ∂L ∂ 2 q ∂L ∂q dt = ∂ q ∂α∂t ˙ ∂ q ∂α ˙ 2 2 − 1 1 ∂q d ∂L ( )dt ∂α dt ∂ q ˙ This first term on the right is zero because the condition exists that all the varied curves pass through the fixed end points and thus the partial derivative of q wrt to α at x1 and x2 vanish. This requires integration by parts twice. the variations δqi are independent and we can apply the calculus of variations lemma. The generalized 2 force of constraint is the Lagrange multipliers term that is added to the original form of Lagrange's equations. the original Lagrangian can be obtained. It's interesting to notice that if the familiar Lagrangian for a simple harmonic oscillator (SHO) plus an extra term is used. L = LSHO + L= d mq q ˙ (−) dt 2 kq 2 d mq q ˙ mq 2 ˙ − + (−) 2 2 dt 2 kq 2 mq q mq 2 ¨ ˙ mq 2 ˙ − − − 2 2 2 2 L=− mq q kq 2 ¨ − 2 2 L= ˙ d This extra term. 1 k ¨ L = − mq q − q 2 2 2 yields ∂L 1 = − mq − kq ∂q 2 − d ∂L =0 dt ∂ q ˙ d d 1 d 1 1 d2 ∂L = ((− mq)) = (− mq) = − m¨ ˙ q dt2 ∂ q ¨ dt dt 2 dt 2 2 Adding them up: −m¨ − kq = 0 q This is interesting because this equation of motion is just Hooke's Law. This crazy looking Lagrangian yields the same equation for simple harmonic motion using the 'jerky' form of Lagrangian's equations. 8 .Applying this result to the Lagrangian. dt (− mqq) probably represents constraint. The particle will eventually fall off but while its on the hoop. Solving for the motion: d ∂L = m¨ r dt ∂ r ˙ ∂L ˙ = mrθ2 − mg cos θ ∂r ∂f ∂r λ =λ ∂r thus ¨ −m¨ + mrθ2 − mg cos θ + λ = 0 r solving for the other equation of motion. Using Lagrange's equations with undetermined multipliers. 2. a. (at the top of the hoop) potential energy is mgr.Homework 3: # 2. Answer: The Lagrangian is 1 ˙ m(r2 + r2 θ2) − mgr cos θ ˙ 2 Where r is the distance the particle is away from the center of the hoop.13 A heavy particle is placed at the top of a vertical hoop. and when θ = 90o (at half of the hoop) potential energy is zero. f = r = a as long as the particle is touching the hoop. 2004 2. Find the height at which the particle falls off. Calculate the reaction of the hoop on the particle by means of the Lagrange's undetermined multipliers and Lagrange's equations. This will be the constraint on the particle. 1 . L=T −V = L= ∂L d ∂L − + ∂qj dt ∂ qj ˙ λ k ∂fk =0 ∂qj with our equation of constraint.14 Michael Good Sept 10. r will equal the radius of the hoop. Here when θ = 0.13. θ = 0 and θ = 0 at t = 0 so. the equations of motion become.d ∂L ˙ = mr2 θ + 2mrrθ ˙ ˙ dt ∂ θ ∂L = mgr sin θ ∂θ ∂fθ λ =λ∗0 ∂θ thus ¨ −mr2 θ − 2mrrθ + mgr sin θ = 0 ˙ ˙ The equations of motion together are: ˙ −m¨ + mrθ2 − mg cos θ + λ = 0 r ¨ ˙ ˙ −mr2 θ − 2mrrθ + mgr sin θ = 0 To find the height at which the particle drops off. ˙ maθ2 − mg cos θ + λ = 0 ¨ −ma2 θ + mga sin θ = 0 Solving for θ. or by ˙ ˙ ˙ dθ dθ dθ d dθ ˙ dθ ¨ = = =θ θ= dt dt dt dθ dt dθ g ˙ ˙ sin θdθ = θdθ a ˙ g θ2 − cos θ = + constant a 2 The constant is easily found because at the top of the hoop. 2 . The force of constraint is λ and λ = 0 when the particle is no longer under the influence of the force of the hoop. the m's cancel and 1 a cancels. we are left with g ¨ sin θ = θ a solving this and noting that ˙ ˙ ˙ θdθ = θdθ by the 'conservation of dots' law Engel has mentioned :). λ can be found in terms of θ. With the angle we can find the height above the ground or above the center of the hoop that the particle stops maintaining contact with the hoop. So finding λ in terms of θ and setting λ to zero will give us the magic angle that the particle falls off. With the constraint. If we say the hoop is a fully circular and somehow fixed with the origin at the bottom of the hoop. The only external force is that of gravity. If the smaller cylinder starts rolling from rest on top of the bigger cylinder. use the method of Lagrange multipliers to find the point at which the hoop falls off the cylinder. The second one comes from no slipping: rφ = s → s = (R + r)θ rφ − rθ = Rθ 3 .2o 3 And if our origin is at the center of the hope. Answer: Two equations of constraint: ρ=r+R r(φ − θ) = Rθ My generalized coordinates are ρ.2g 2g ˙ cos θ + = θ2 a a Plug this into our first equation of motion to get an equation dependent only on θ and λ − ma[− 2g 2g cos θ +] − mg cos θ = −λ a a −3mg cos θ + 2mg = −λ Setting λ = 0 because this is at the point where the particle feels no force from the hoop. then the height that it stops touching the hoop is just R cos θ0 or 2 2 h = R cos(cos−1) = R 3 3 . The first equation comes from the fact that as long as the hoop is touching the cylinder the center of mass of the hoop is exactly r + R away from the center of the cylinder. θ. and θ0 equals 2 θ0 = cos−1 ( ) = 48.14 A uniform hoop of mass m and radius r rolls without slipping on a fixed cylinder of radius R as shown in figure. I'm calling it f1 . then we have just moved down by R and the new height is 2 5 H =R+ R= R 3 3 2. and φ. I'm calling this equation f2 . So I'm going to apply the constraints to my equations of motion. This will tell me the value of θ.r(φ − θ) = Rθ Where θ is the angle ρ makes with the vertical and φ is the angle r makes with the vertical. The constraints tell me: 4 . f1 = ρ − r − R = 0 f2 = Rθ − rφ + rθ = 0 The Lagrangian is T − V where T is the kinetic energy of the hoop about the cylinder and the kinetic energy of the hoop about its center of mass. attempt to get an equation for θ. and then set λ1 equal to zero because that will be when the force of the cylinder on the hoop is zero. The potential energy is the height above the center of the cylinder. This will tell me the point that the hoop drops off the cylinder. Therefore m 2 ˙ ˙ (ρ + ρ2 θ2 + r2 φ2) − mgρ cos θ ˙ 2 Solving for the equations of motion: L= d ∂L ∂L − = dt ∂ ρ ˙ ∂ρ λk k ∂fk ∂ρ ∂f1 ∂f2 ¨ m¨ − mρθ2 + mg cos θ = λ1 ρ + λ2 ∂ρ ∂ρ ¨ m¨ − mρθ2 + mg cos θ = λ1 ρ d ∂L ∂L − = ˙ dt ∂ θ ∂θ λk k (1) ∂fk ∂θ d ∂f1 ∂f2 ˙ (mρ2 θ) − (−mgρ sin θ) = λ1 + λ2 dt ∂θ ∂θ ¨ ˙ ˙ mρ2 θ + θ2mρρ + mgρ sin θ = λ1 (0) + λ2 (R + r) ¨ mρ2 θ + 2mρρθ + mgρ sin θ = λ2 (R + r) ˙ ˙ d ∂L ∂L = − ˙ dt ∂ φ ∂φ λk k (2) ∂fk ∂φ d ∂f1 ∂f2 ˙ (mr2 φ) − 0 = λ1 + λ2 dt ∂φ ∂φ ¨ mr2 φ = −λ2 r (3) I want the angle θ. Looking for an equation in terms of only θ and λ1 will put me in the right position. ˙ ˙ 2θθ = −B sin θθ B ˙ θ = − sin θ 2 Thus B=− q R+r ˙ From initial conditions. θ = 0 at t = 0 we have A: A = −B → A= q R+r 5 . θ = 0. λ2 m(R + r) (4) ¨ m(R + r)θ + mg sin θ = λ2 ¨ λ2 − mg sin θ θ= m(R + r) Setting (4) = (5) −λ2 = λ2 − mg sin θ mg sin θ λ2 = 2 Plugging (6) into (4) yields a differential equation for θ ¨ θ= −g sin θ 2(R + r) (5) (6) ˙ If I solve this for θ2 I can place it in equation of motion (1) and have an expression in terms of θ and λ1 .ρ=r+R φ= ρ=ρ=0 ˙ ¨ → ¨ R + rθ ¨ φ= r R+r R+r ˙ ¨ θ → φ= θ r r Solving (3) using the constraints. ¨ mr2 φ = −λ2 r ¨ θ=− Solving (2) using the constraints. This differential equation can be solved by trying this: ˙ θ2 = A + B cos θ Taking the derivative. the height that the center of mass of the hoop falls off is cos θ0 = 1 ρ 2 Or if you prefer the height that the hoop's surface stops contact with cylinder: 1 h= R 2 .Therefore ˙ θ2 = q q − cos θ R+r R+r Now we are in a position to plug this into equation of motion (1) and have the equation in terms of θ and λ1 −m(R + r)(q q − cos θ) + mg cos θ = λ1 R+r R+r −mg + 2mg cos θ = λ1 mg(2 cos θ − 1) = λ1 Setting the force of constraint equal to zero will give us the angle that the hoop no longer feels a force from the cylinder: 2 cos θ0 − 1 = 0 1 → θ0 = 60o 2 With our origin at the center of the cylinder. hcm = ρ cos(60o) = 6 . Obtain the Lagrange equations of motion assuming the only external forces arise from gravity.13. h. The equations of motion are then: L= d ∂L ∂L − =0 ˙ dt ∂ θ ∂θ ¨ ma2 θ = ma2 ω 2 sin θ cos θ + mga sin θ We see that the Lagrangian does not explicitly depend on time therefore the energy function. and the potential energy is considered negative at the bottom of the hoop. 3. w is constant as well. and a is the radius. What is the value of ω0 ? Answer: To obtain the equations of motion. My θ is the angle from the z-axis.18 A point mass is constrained to move on a massless hoop of radius a fixed in a vertical plane that rotates about its vertical symmetry axis with constant angular speed ω.20 Michael Good Sept 20. We only need one generalized coordinate. there can be a solution in which the particle remains stationary on the hoop at a point other than the bottom. the only stationary point for the particle is at the bottom of the hoop. 3. because the radius of the hoop is constant. What are the constants of motion? Show that if ω is greater than a critical value ω0 .18. we need to find the Lagrangian. and the point mass is constrained to this radius.14. but if ω < ω0 . and zero where the vertical is at the center of the hoop.21. 3. 2. ˙ ∂L − L h=θ ˙ ∂θ ˙ ˙ 1 ˙ h = θma2 θ − ma2 (θ2 + ω 2 sin2 θ) − mga cos θ 2 1 . 2004 2. 1 ˙ ma2 (θ2 + ω 2 sin2 θ) − mga cos θ 2 Where the kinetic energy is found by spherical symmetry. is conserved (Goldstein page 61). while the angular velocity.Homework 4: # 2. If we speed up this hoop.This simplifies to: 1 1 ˙ ma2 θ − (ma2 ω 2 sin2 θ − mga cos θ) 2 2 Because the 'energy function' has an identical value to the Hamiltonian. So the point mass moves up the hoop. the bottom. to a nice place where it is swung around and maintains a stationary orbit. h= 1 Vef f = mga cos θ − ma2 ω 2 sin2 θ 2 The partial of Vef f with respect to θ set equal to zero should give us a stationary point. θ=0 θ=π θ = arccos(− g) aω 2 At the top. θ = π is stable. and is the only stationary point for the particle. ω0 = g a The top of the hoop is unstable. and some angle that suggests a critical value of ω. θ = arccos(− 2 . ω > ω0 . the bottom. our angle 2 ω0) ω2 is stable and θ = π becomes unstable. Therefore anything ω < ω0 . the only stable minimum is at θ = π. the effective potential is the second term. but at the bottom we have a different story. If I set ω = ω0 and graph the potential. ∂Vef f = mga sin θ + ma2 ω 2 sin θ cos θ = 0 ∂θ ma sin θ(g + aω 2 cos θ) = 0 This yields three values for θ to obtain a stationary point. of force constant k and zero equilibrium length. 2 V (r. so I'll need a pair of transformation equations relating the two frames. y) so as to write the stubborn potential energy in terms of the lab frame is done with some algebra. l) = The energy needs to be written down fully in one frame or the other. I'll use (r. On the carriage. by drawing a diagram. • What is the energy of the system? Is it conserved? • Using generalized coordinates in the laboratory system. l). held by a spring fixed on the beam. rails. and its potential energy is easy to write down.2. relating (x. with a constant angular speed ω. In the rotating frame. whose other end is fixed on the beam. Answer: Energy of the system is found by the addition of kinetic and potential parts. and carriage are assumed to have zero mass. The carriage is attached to one end of a spring of equilibrium length r0 and force constant k. what is the Lagrangian? What is the Jacobi integral? Is it conserved? Discuss the relationship between the two Jacobi integrals. Since the small spring has zero equilibrium length. then the potential energy for it is just 1 kl2 . using Cartesian coordinates is 1 m(x2 + y 2) ˙ ˙ 2 Potential energy is harder to write in lab frame. y) to (r. 3 . Solving for them. y). springs. The length of the second spring is at all times considered small compared to r0 . The potential. another set of rails is perpendicular to the first along which a particle of mass m moves. as shown in the figure below. The whole system is forced to move in a plane about the point of attachment of the first spring. y) and l(x. yields x = (r0 + r) cos ωt − l sin ωt y = (r0 + r) sin ωt + l cos ωt Manipulating these so I may find r(x. what is the Jacobi integral for the system? Is it conserved? • In terms of the generalized coordinates relative to a system rotating with the angular speed ω. in the rotating frame is T (x. in the lab frame. the system looks stationary. That is. y) = 1 k(r2 + l2) 2 Where r is simply the distance stretched from equilibrium for the large spring. l) to denote the rotating frame coordinates. The kinetic. (x. Beam.21 A carriage runs along rails on a rigid beam. adding the two equations and solving for r yields r = x cos ωt + y sin ωt − r0 Multiplying x by sin and y by cos. l) = Thus 1 1 ˙ m(ω 2 (r0 + r +)2 + (r − lω)2) ˙ 2 ω 4 . To find E(r.T. y) = This energy is explicitly dependent on time. are C. l) we are lucky to have an easy potential energy term. y) = 2 (x2 + y 2) We need 1 E(r. T (x.Multiplying x by cos ωt and y by sin ωt. Taking derivatives of ˙ ˙ x and y yield x = −ω(r0 + r) sin ωt + r cos ωt − lω cos ωt − l ˙ sin ωt ˙ y = ω(r0 + r) cos ωt + r sin ωt − lω sin ωt − l ˙ cos ωt ˙ Squaring both and adding them yields x2 + y 2 = ω 2 (r0 + r)2 r2 + l2 ω 2 + l˙2 + C. C. adding and solving for l yields l = −x sin ωt + y cos ωt Plugging these values into the potential energy to express it in terms of the lab frame leaves us with 1 1 m(x2 + y 2) + k((x cos ωt + y sin ωt − r0)2 + (−x sin ωt + y cos ωt)2) ˙ ˙ 2 2 E(x. E(x. ˙ ˙ Where cross terms. y) is not conserved. = 2ω(r0 + r)l˙ − 2rlω ˙ For kinetic energy we know have 1 m(ω 2 (r0 + r)2 r2 + l2 ω 2 + l˙2 + 2ω(r0 + r)l˙ − 2rlω) ˙ ˙ 2 Collecting terms T (r. In the rotating frame this may be a different story.T. l) + k(l2 + r2) 2 1 Where in the laboratory frame. l) = T (r. Thus it is NOT conserved in the lab frame. l) = T (r. but now our kinetic energy is giving us problems.T. y) ˙ ˙ 2 V (x. Bringing it ˙ ˙ together 1 1 m(x2 + y 2) + k((x cos ωt + y sin ωt − r0)2 + (−x sin ωt + y cos ωt)2) ˙ ˙ 2 2 h= This is equal to the energy. h(x. l) = In the laboratory frame. y) = The Jacobi integral. y) ˙ ˙ ˙ ˙ 2 Notice that V (x. therefore energy in the rotating frame is conserved. l) − k(r2 + l2) 2 5 . d ∂L h=− =0 dt ∂t we know h(x. y) = E(x. the Lagrangian is 1 L(r. L(x. For the rotating frame. y) is not conserved in the lab frame. h ∂x ˙ ∂y ˙ 1 ˙ ˙ h = xmx + ymy − m(x2 + y 2) + V (x. or energy function is h= i qi ˙ ∂L ∂ qi −L ˙ We have h=x ˙ ∂L ∂L +y ˙ − L(x. E(r. l) = T (r. y) = Where 1 k((x cos ωt + y sin ωt − r0)2 + (−x sin ωt + y cos ωt)2) 2 1 m(x2 + y 2) − V (x. E(r. y) Because it is explicitly dependent on time. y) does not have any dependence on x or y. l) is conserved. y).l˙ 1 1 m(ω 2 (r0 + r +)2 + (r − lω)2) + k(l2 + r2) ˙ 2 ω 2 This has no explicit time dependence. the Lagrangian is just T (x. y) − V (x. It is of the from E = − 2 Iω 2 . 6 . or Jacobi integral is T (r. l) = d ∂L h=− =0 dt ∂t We have h(r. l) ˙ ˙ ω Collecting terms. and this nice way of writing it reveals an energy term of rotation in the lab frame that can't be seen in the rotating 1 frame. 1 1 1 ˙ ˙ m(r2 +l˙2)+ k(l2 +r2)+ [rmlω−lωmr−ml2 ω 2 +l˙mω 2 (r0 +r)2 −mω l(r0 +r)] ˙ ˙ ˙ 2 2 2 h= Yields 1 1 1 m(r2 + l˙2) + k(l2 + r2) − mω 2 (l2 + (r0 + r)2) ˙ 2 2 2 This has no time dependence. h(r.Where 1 l ˙ m(ω 2 (r0 + r +)2 + (r − lω)2) ˙ 2 ω The energy function. l) conserved in the rotating frame. l) = h(r. l) +l ∂r ˙ ∂ l˙ l˙ ˙ h(r. l) = rm(r − lω) + lmω(r0 + r +) − L(r. l) = r ˙ ∂L ˙ ∂L − L(r. l) = r with some heavy algebra 1 l˙ 1 ˙ 1 h = (r−lω)(mr− m(r−lω))+(r0 +r+)(mω l− mω 2 (r0 +r+)))+ k(l2 +r2) ˙ ˙ ˙ 2 ω 2 ω 2 l˙ 1 1 1 mr 1 ˙ + mlω) + (r0 + r +)(mω l ˙ − mω 2 (r0 + r)) + k(l2 + r2) 2 2 ω 2 2 ω 2 h = (r − lω)(˙ More algebraic manipulation in order to get terms that look like kinetic energy. 34) in Goldstein. u= 1 sin θ d u= [− cos−2 θ(− sin θ)] = dθ 2R 2R cos2 θ The derivative of this is d sin θ 1 = [sin θ(−2 cos−3 θ)(− sin θ) + cos−2 θ cos θ] dθ 2R cos2 θ 2R Thus 1 2 sin2 θ cos2 θ 1 + sin2 θ d2 u= [+]= dθ2 2R cos3 θ cos3 θ 2R cos3 θ 2R cos3 θ d2 1 + sin2 θ cos2 θ 2 u+u= + = 2 3θ dθ 2R cos 2R cos3 θ 2R cos3 θ That is 8R2 = 8R2 u3 cos3 θ 8R3 1 Solving for V (u) by integrating yields. Answer: Using the differential equation of the orbit. y and v as a function of angle around the circle and show that ˙ ˙ all three quantities are infinite as the particle goes through the center of force. 7 . d2 1 m d V( ) u+u=− 2 2 dθ l du u Where r = 1/u and with the origin at a point on the circle. • Show that for orbit described the total energy of the particle is zero.3. • Find x. equation (3. then the force varies as the inverse-fifth power of the distance. • Find the period of the motion.13 • Show that if a particle describes a circular orbit under the influence of an attractive central force directed toward a point on the circle. a triangle drawn with r being the distance the mass is away from the origin will reveal r = 2R cos θ 1 2R cos θ Plugging this in and taking the derivative twice. f (r) = − Is the energy zero? Well. we know V (r). 2 2 8 2R2 l2 2R2 l2 − =0 mr4 mr4 → T = 2R2 l2 mr4 . lets find T (r) and hope its the negative of V (r). The period of the motion can be thought of in terms of θ as r spans from θ = − π to θ = π . plugging these in. ˙ l = mr2 θ ˙ l 2 = m2 r 4 θ 2 ˙ T = 2mR2 θ2 Which shows that E =T +V = the total energy is zero. T = Where r = 2R cos θ r2 = 4R2 cos2 θ So. l.1 8R2 l2 4 V( )=− u u 4m and we have V (r) = − with force equal to d 8l2 R2 V (r) = − dr mr5 This force is inversely proportional to r5 . T = 1 ˙ ˙ m(4R2 sin2 θθ2 + 4R2 cos2 θθ2) 2 T = → ˙ r = −2R sin θθ ˙ ˙ r2 = 4R2 sin2 θ2 ˙ 1 ˙ m(r2 + r2 θ2) ˙ 2 2l2 R2 mr4 ˙ m4Rθ2 ˙ = 2mR2 θ2 2 Put this in terms of angular momentum. Remembering that r = 2R cos θ. π 2 π 2 r2 dθ −π 2 4R2 cos2 θdθ −π 2 π 2 −π 2 4mR2 θ 1 cos θdθ= (+ sin 2θ l 2 4 2)= −π 2 4mR2 π π (+) l 4 4 2mπR2 l For x. y.π 2 π 2 P = −π 2 dt = −π 2 dt dθ dθ This is π 2 P = −π 2 dθ ˙ θ ˙ Because θ = l/mr2 in terms of angular momentum. and v as a function of angle. ˙ x = −4R cos θ sin θ = −2Rθ sin 2θ ˙ ˙ y = 2Rθ cos 2θ ˙ v= ˙ x2 + y 2 = 2Rθ ˙ ˙ What is θ? In terms of angular momentum we remember 9 . P = x = r cos θ = 2R cos2 θ y = r sin θ = 2R cos θ sin θ = R sin 2θ Finding their derivatives. we have π 2 P = −π 2 mr2 dθ l π 2 m P = l From above we have r2 m P = l 4mR2 P = l And finally. it can be shown that all three quantities ˙ ˙ are infinite as particle goes through the center of force. • Prove that in the same central force as above.˙ l = mr2 θ ˙ Plugging in our r. θ becomes close to ± π . show that perihelion distance of the parabola is one-half the radius of the circle. Goldstein equation 3. y and v are directly proportional to the θ term. 2 θ = ±(Note that as δ→0 θ→± π 2 ˙ θ→∞ π − δ) 2 ˙ θ= l 4mR2 ˙ All x.55. 1 mk = 2 [1 + cos(θ − θ)] r l we have for the circle. the speed of a particle at √ any point in a parabolic orbit is 2 times the speed in a circular orbit passing through the same point. =0 1 mk l2 = 2 → rc = rc l mk For the parabola. The x may be ˙ ˙ questionable at first because it has a sin 2θ and when sin 2θ → 0 as θ → π/2 we ˙ may be left with ∞ ∗ 0. =1 1 mk l2 = 2 (1 + 1) → rp = rp l 2mk 10 π 2 . But looking closely at θ we can tell that x= ˙ l −4Rl cos θ sin θ =− tan θ 4mR2 cos2 θ mR tan θ → ∞ as θ → ± 2.14 • For circular and parabolic orbits in an attractive 1/r potential having the same angular momentum. and solving for θ cos2 θ As we got closer to the origin. Answer: Using the equation of orbit. and not forgetting that k = l2 /mr. l2 l2 ˙ → θ2 = 2 4 r= mk(1 + cos θ) m r we have 2k 2r2 l2 mkr 2 → vp = m2 r4 l2 mr For the speed of the parabola.55). ˙ l2 l2 θ d ( )= sin θ dt mk(1 + cos θ) mk(1 + cos θ)2 2 ˙ vp = r2 θ2 (sin2 θ + 1) (1 + cos θ)2 2 + 2 cos θ) (1 + cos θ)2 2 ˙ vp = r2 θ2 (2 vp = ˙ 2r2 θ2 1 + cos θ Using r for a parabola from Goldstein's (3. we then have 2 vp = vp = Thus √ 2 k mr vp = √ 2vc 11 .So rc 2 The speed of a particle in a circular orbit is rp = 2 ˙ vc = r2 θ2 → 2 vc = r2 (l2) m2 r 4 → vc = l mr In terms of k. this is equal to √ mrk k l = = mr mr mr The speed of a particle in a parabola can be found by 2 ˙ vp = r2 + r2 θ2 ˙ r= ˙ Solving for vp . A uniform distribution of dust in the solar system adds to the gravitational attraction of the Sun on a planet an additional force F = −mCr where m is the mas of the planet.20. • Show that nearly circular orbits can be approximated by a precessing ellipse and find the precession frequency. Is the precession in the same or opposite direction to the orbital angular velocity? Answer: The equation for period is T = For a circular orbit.58): k l2 = 2 3 r0 mr0 In our case. C is a constant proportional to the gravitational constant and the density of the dust. ˙ θ= Thus T = Goldstein's equation after (3. This additional force is very small compared to the direct Sun-planet gravitational force. • Calculate the period of radial oscillations for slight disturbances from the circular orbit. mCr0 + Solving for l yields l= 4 mr0 k + mr2 Cr0 2π ˙ θ l mr2 2πmr2 l k l2 2 = mr 3 r0 0 12 . we have an added force due to the dust. and r is the radius vector from the Sun to the planet(both considered as points). • Calculate the period for a circular orbit of radius r0 of the planet in this combined field. we would have C = 0 and our period would be ωorb = T0 = 2π k 2 mr0 → √ ω0 = mrk. Dividing our orbital period by β will give us the period of the oscillations. β is the number of cycles of oscillation that the particle goes through in one complete orbit.Plugging this in to our period.46) u≡ β2 = 3 + Solve this with f = mCr + k/r2 df 2k = − 3 + mC dr r β2 = 3 + r β2 = Now k r 2 + 4mCr k r 2 + mCr r df f dr r=r0 − 2k + mC r3 k r2 + mCr β2 = k mr 3 + 4C k mr 3 + C → 13 . T = 2πmr2 mr0 k + 4 m2 Cr0 → T = 2π k 3 mr0 +C Here the orbital angular velocity is k +C mr3 This is nice because if the dust was not there.45) in Goldstein page 90. states that for small deviations from circularity conditions. k 2 mr0 2 which agrees with l = mr0 ω0 and l = The period of radial oscillations for slight disturbances from the circular orbit can be calculated by finding β. Tosc = T β Equation (3. 1 = u0 + a cos βθ r Substitution of this into the force law gives equation (3. ωprec = k 2Cmr3 Cmr3 [1 + − (1 +)] = mr3 k 2k k 4Cmr3 − Cmr3 3Cmr3 = 3 mr 2k 2k 1+ Cmr3) k ωprec = k 2Cmr3 Cmr3 [−] mr3 k 2k k 3C = mr3 2 mr3 k ωprec = Therefore. our period of radial oscillations is Tosc = Here k + 4C mr3 A nearly circular orbit can be approximated by a precessing ellipse. I'm going to subtract the orbital angular velocity from the radial angular velocity. 14 . for a nearly circular orbit. To find the precession frequency. ωprec = 4Cmr3 k − (1+ 3 k mr Using the binomial expansion.Tosc = T β β= k mr 3 k mr 3 + 4C +C Therefore. a precessing ellipse will hug closely to the circle that would be made by e = 0. ωprec = ωr − ωorb k k + 4C − +C mr3 mr3 Fixing this up so as to use the binomial expansion. The equation for an elliptical orbit is ωr = r= a(1 − e2) 1 + e cos(θ − θ0 2π k mr 3 + 4C with e a r≤a Show that the scattering produced by such a potential in classical mechanics is identical with the refraction of light rays by a sphere of radius a and relative index of refraction E + V0 E This equivalence demonstrates why it was possible to explain refraction phenomena both by Huygen's waves and by Newton's mechanical corpuscles.2E(s2 + ds = dΘ πk Putting everything in terms of x. s2 + So now...32 A central force potential frequently encountered in nuclear physics is the rectangular well. σ(Θ) = 1−x k 1 1 2E π sin πx (x(2 − x))2 And since we know dΘ = πdx. σ(Θ)dΘ = k (1 − x)dx 2 (2 − x)2 sin πx 2E x 3. k 2E k 3 2 2E) k (1 − x)2 k k 1 k = + = 2E 2E x(2 − x) 2E 2E x(2 − x) s ds σ(Θ) = = sin Θ dΘ √(1−x) x(2−x) sin πx 2E(s2 + πk k 3 2 2E) k 2E √(1−x) x(2−x) = k 1 2E(2E x(2−x)) 2 3 sin πx πk = And this most beautiful expression becomes.. σ(Θ) = 1 1 k 1 2E k 3 ( )2 ( )( )2 sin πx π 2E k 2E 1−x 1 3 x(2 − x) (x(2 − x)) 2 After a bit more algebra. Show also that the differential cross section is n= σ(Θ) = n2 a2 (n cos Θ − 1)(n − cos Θ) 2 2 4 cos Θ (1 + n2 − 2n cos Θ)2 2 2 What is the total cross section? 4 . then putting our total angle scattered. σ(Θ) = Θ = 2(θ1 − θ2) This is because the light is refracted from its horizontal direction twice. sds sin ΘdΘ If the scattering is the same as light refracted from a sphere.Answer: Ignoring the first part of the problem. We know sin θ1 = s/a and using Snell's law. after hitting the sphere and leaving the sphere. in terms of the angle of incidence and transmission. Solve for sin Θ and cos Θ in terms of s 2 2 sin Θ s s s s s s = sin(arcsin −arcsin) = sin arcsin cos arcsin −cos arcsin sin arcsin 2 a na a na a na This is s s2 s2 s cos(arccos 1 − 2 2) − cos(arccos(1 − 2) a n a a n a √ Using arcsin x = arccos 1 − x2 and sin(a − b) = sin a cos b − cos a sin b. Θ. Where θ1 − θ2 is the angle south of east for one refraction. and just solving for the differential cross section. to solve for s2 and then ds2 /dΘ and solve for the cross section via σ= 1 ds2 1 sds ds2 = = sin ΘdΘ 2 sin Θ dΘ 4 sin Θ cos Θ dΘ 2 2 Here goes. we know n= s sin θ1 → sin θ2 = sin θ2 na s s − arcsin) a na Expressing Θ in terms of just s and a we have Θ = 2(arcsin Now the plan is. Now we have = sin Θ s = (n2 a2 − s2) − 2na2 a2 − s2) Doing the same thing for cos Θ yields 2 5 . we can now do some calculus. ds2 a2 sin Qn2 = [cos Q(1 − 2n cos Q + n2) − n sin2 Q] dΘ q2 ds2 n2 a2 = 2 sin Q[cos Q − 2n cos2 Q + n2 cos Q − n(1 − cos2 Q)] dΘ q Expand and collect n2 a2 ds2 = 2 sin Q[−n cos2 Q + cos Q + n2 cos Q − n] dΘ q Group it up ds2 n2 a2 = 2 sin Q(n cos Q − 1)(n − cos Q) dΘ q Plug back in for Q and q 2 : 6 . Also to save space. I 2 like using the letter q.cos Θ 1 (a2 − s2 = 2 na2 n2 a2 − s2 + s2) Using cos(a − b) = cos a cos b + sin a sin b. I'm going to let q 2 equal the denominator squared. lets say Θ = Q. Still solving for s2 in terms of cos and sin's we proceed sin2 This is sin2 Note that n2 a2 − s2 So we have s2 2s2 Θ 2s2 s2 Θ Θ = 2 2 (n2 + 1 − 2 − 2n cos + 2) = 2 2 (1 + n2 − 2n cos) 2 n a a 2 a n a 2 a2 − s2 = na2 cos Θ − s2 2 2s2 s2 2s4 (n2 + 1) − 2 4 − 2 4 2 a2 n n a n a n2 a2 − s2 a2 − s2 s2 Θ = 2 4 (n2 a2 − s2 − 2 n2 a2 − s2 2 n a a2 − s2 + a2 − s2) sin2 Solving for s2 = n2 a2 sin2 Θ 2 1 + n2 − 2n cos Θ 2 Glad that mess is over with. Making a partial substitution to see where to go: qmin = 1 − 2 + n2 = n2 − 1 (n−1)2 qmax = n2 − 2n + 1 = (n − 1)2 (n−1)2 n2 −1 σT = n2 −1 2πa2 n2 (nx − 1)(n − x) dq = πa2 q2 −2n 7 −n(nx − 1)(n − x) dq q2 . Θ we will find it easier to plug in x = cos 2 as a substitution. If s > a. let q equal the term in the denominator. At s = a we have maximum Θ. This integral is still hard to manage. sin Θ = 2 sin Θ cos Θ was used on the sin Θ. we'll have Θmax . 1 σT = π 1 n a2 n2 (nx − 1)(n − x) 2dx (1 − 2nx + n2)2 1 Θ Θmax 1 dx = − sin dΘ cos = 2 2 2 n The half angle formula. this time. so make another substitution. When (n cos Θ − 1) is zero. 1 4 sin Θ 2 σ= cos Θ 2 n2 a2 sin Θ (n cos Θ − 1)(n − cos Θ) ds2 1 2 2 2 = Θ Θ dΘ 4 sin 2 cos 2 (1 − 2n cos Θ + n2)2 2 We obtain σ(Θ) = n2 a2 (n cos Θ − 1)(n − cos Θ) 1 2 2 4 cos Θ (1 − 2n cos Θ + n2)2 2 2 The total cross section involves an algebraic intensive integral. to simplify our integral. The total cross section is given by Θmax σT = 2π 0 σ(Θ) sin ΘdΘ To find Θmax we look for when the cross section becomes zero. its as if the incoming particle 2 1 misses the 'sphere'. the 2 2 negative sign switched the direction of integration.n2 a2 sin Θ (n cos Θ − 1)(n − cos Θ) ds2 2 2 2 = dΘ (1 − 2n cos Θ + n2)2 2 Using our plan from above. and the factor of 2 had to be thrown in to make the dx substitution. q = 1 − 2nx + n2 → dq = −2ndx where The algebra must be done carefully here. So using Θmax = 2 arccos n . subtract a 1. That is: q 2 − (n2 − 1)2 q 2 + n4 + 2n2 − 1 = = −n(nx − 1)(n − x) 4 4 Now. add a 2n2 and divide the whole thing by 4 we'll get the above numerator.Expanding q 2 to see what it gives so we can put the numerator in the above integral in terms of q we see q 2 = n4 + 1 + 2n2 − 4n3 x − 4nx + 4n2 x2 Expanding the numerator −n(nx − 1)(n − x) = −n3 x − nx + n2 x2 + n2 If we take q 2 and subtract a n4 . our integral is (n−1)2 σT = πa 2 n2 −1 q 2 − (n2 − 1)2 dq 4q 2 This is finally an integral that can be done by hand πa2 σT = 4 (n2 − 1)2 πa2 (n2 − 1)2 1− dq = (z + q2 4 z (n−1)2) n2 −1 After working out the few steps of algebra. πa2 4n2 − 8n + 4 = πa2 4 n2 − 2n + 1 The total cross section is σT = πa2 8 . 1. 2004 4. are A(BC) = k Aik (m Bkm Cmj) (AB)C = m (k Aik Bkm)Cmj Both the elements are the same. σx σx = ˜ σz σz = ˜ 0 1 1 0 0 1 1 0 1 0 0 −1 = = 1 0 0 1 1 0 0 1 =I =I 1 0 0 −1 The product of these two: σx σ z = 0 1 1 0 1 0 0 −1 = 0 −1 1 0 ≡q 1 . As long as the products are defined. 4. 4. They only differ in the order of addition.14.10.15 Michael Good Oct 4.1 Prove that matrix multiplication is associative. and there are finite dimensions. σx . matrix multiplication is associative. and σz are both orthogonal. Answer: Matrix associativity means A(BC) = (AB)C The elements for any row i and column j. Orthogonality may be defined by AA = I The Pauli spin matrices.Homework 6: # 4.2. 4. 2 Prove the following properties of the transposed and adjoint matrices: AB = B A (AB)† = B † A† Answer: For transposed matrices AB = AB ij = ABji = ajs bsi = bsi ajs = Bis Asj = (B A)ij = B A As for the complex conjugate.r bsi aks akr brj = ˜ bsi (AA)sr brj This is ABAB = ˜ bsi δsr brj = BBij = δij Therefore the whole matrix is I and the product ABAB = I is orthogonal. if AA = 1 BB = 1 0 −1 1 0 0 −1 1 0 = 1 0 0 1 =I then both A.s. 4.s. We can look at ABAB = k (AB)ik (AB)kj = k AB ki AB kj = k. (AB)† = (AB)∗ From our above answer for transposed matrices we can say AB = B A 2 . and B are orthogonal.is also orthogonal: qq = ˜ More generally.r aks bsi akr brj The elements are aks bsi akr brj = k. A ≡ eB = 1 + B + B 2 + . we can provide a rigorous proof. + 2 n! then prove the following properties: • eB eC = eB+C . noting that i + j = k k 0 BiC j = i!j! 3 k 0 B k−j C j (k − j)!j! ..10 If B is a square matrix and A is the exponential of B. • A−1 = e−B • eCBC −1 = CAC −1 • A is orthogonal if B is antisymmetric Answer: Providing that B and C commute. Looking at the kth order terms.. providing B and C commute... BC − CB = 0 we can get an idea of what happens: C2 C2 B2 B2 +O(B 3))(1+C + +O(C 3)) = 1+C + +B +BC + +O(3) 2 2 2 2 BC = CB (1+B + This is 1 (B + C)2 1+(B +C)+ (C 2 +2BC +B 2)+O(3) = 1+(B +C)+ +O(3) = eB+C 2 2 Because BC = CB and where O(3) are higher order terms with products of 3 or more matrices. defined by the infinite series expansion of the exponential. Bn 1 + . by using the expansion for exp we get. Expanding the left hand side of eB eC = eB+C and looking at the kth order term.And so we have (AB)† = (AB)∗ = (B A)∗ = B ∗ A∗ = B † A† 4.. . Bence): (B + C)k 1 = k! k! k 0 k! B k−j C j = (k − j)!j! k 0 B k−j C j (k − j)!j! we get the same term. (a proof of which is given in Riley..... A−1 = e−B To prove eCBC its best to expand the exp ∞ −1 = CAC −1 0 1 CBC −1 CBC −1 CBC −1 CBC −1 CBC −1 .. Hobsen. To prove A−1 = e−B We remember that A−1 A = 1 and throw e−B on the right A−1 Ae−B = 1e−B A−1 eB e−B = e−B and from our above proof we know eB eC = eB+C so A−1 eB−B = e−B Presto.+ +. n! 2 n! Do you see how the middle C −1 C terms cancel out? And how they cancel each out n times? So we are left with just the C and C −1 on the outside of the B's. QED. ∞ 0 1 (CBC −1)n = n! B ∞ 0 1 CB n C −1 = CeB C −1 n! Remember A = e and we therefore have eCBC −1 = CAC −1 4 . (CBC −1)n = 1+CBC −1 + +.and using the binomial expansion on the right hand side for the kth order term.. For any other set of i. so A = A−1 and we can happily say A is orthogonal.14 • Verify that the permutation symbol satisfies the following identity in terms of Kronecker delta symbols: ijp rmp = δir δjm − δim δjr • Show that ijp ijk = 2δpk Answer: To verify this first identity. 4. we know that e−B = A−1 .To prove A is orthogonal A = A−1 if B is antisymmetric −B = B We can look at the transpose of A ∞ A= 0 Bn = n! ∞ 0 Bn = n! ∞ 0 (−B)n = e−B n! But from our second proof. If ijp j=m=i j=m=i ijp = rmp and whether or not is ±1 the product of the two gives i=m j=r=i 5 . if i=r then a +1.e. all we have to do is look at the two sides of the equation. if the right hand side has i=r we get +1. For the left hand side. i. analyzing the possibilities. If i=m j=r=i we get −1. and m we get 0. j. r. lets match conditions. we set i = r. this is called 'contracting' we get ijk ijp = δjj δkp − δjp δkj Using the summation convention. cast in a different form: ijk imp = δjm δkp − δjp δkm This is equivalent because the product of two Levi-Civita symbols is found from the deteriment of a matrix of delta's. z). its helpful to label the axes of rotation ˙ ˙ ˙ for θ. none can have the same value as p. (not all four subscripts may be equal because then it would be = 0 as if i = j or r = m). j. To show that ijp ijk = 2δpk we can use our previous identity. If we also set j = m. that any of the subscripts may take. z) we can find the angular velocity along the space axes (x.15 Show that the components of the angular velocity along the space set of axes are given in terms of the Euler angles by ˙ ˙ ωx = θ cos φ + ψ sin θ sin φ ˙ ˙ ωy = θ sin φ − ψ sin θ cos φ ˙ ˙ ωz = ψ cos θ + φ Answer: Using the same analysis that Goldstein gives to find the angular velocity along the body axes (x . y.then rmp = equal to −1. ψ and φ. m. r. that is = δir δjm δkp + δim δjp δkr + δip δjr δkm − δim δjr δkp − δir δjp δkm − δip δjm δkr ijk rmp For our different form. δjj = 3. To make a drawing easier. 6 . the only non-zero values are the ones above. Since there are only three values. y . jip =− ijp and whether or not ijp is ±1 the product is now These are the only nonzero values because for i. ijk ijp = 3δkp − δkp = 2δkp ijk ijp 4. Therefore because the line of nodes is perpendicular to the z space axis there is no component of φ contributing to angular velocity around ˙ ˙ the z space axis. ψ revolves around z . We can see that φ just revolves around z in the first place! Now lets look at φ Right? So there is no need to make any 'transformation' or make any changes.N. that is. Does this make sense? We ˙ ˙ find the z part. θz = 0. because they both lie in the same xy plane. which is θ. What is θx ? Well. θ is along the line of nodes. Add them all up for our total ωz . Lets look at ψx . So we ˙ ˙ get after two projections. that is. Yes? So ˙ ˙ ˙ ˙ θx = θ cos φ. See how φ revolves around the z axis? Well. If φ = 0 we would have projected it right on top of the y − axis! So we can make sure that if φ = 0 we have a zero component for x by multiplying by sin φ. ˙ x = 0. Now its in the same plane. That component depends on how much angle there is between z and z. But where is it facing in this plane? We can see that depends on the angle φ. We first have to find the component in the same xy plane. θ is along the line of nodes. ˙ ˙ ˙ ˙ ˙ ωx = θx + ψx + φx = θ cos φ + ψ sin θ sin φ + 0 I'll explain ωy for kicks. What about ψz ? Well. angular velocity in the x space axis. So 7 . ˙ ˙ Lets take φz = φ. which is the adjacent side to θ. Thus we have ψz = ψ cos θ. So to get into ˙ ˙ the xy plane we can take ψx. θ changes and revolves around the line of nodes axis. Now look at ψx .˙ θ → L. Its y component depends on the angle φ. then find the component of the x direction. the z axis is perpendicular to the ˙ x axis there for there is no component of φ that contributes to the x space axis. ψx = ψ sin θ sin φ. Try ωx . Look ˙ ˙ for θy . We can see that ψ is along the z body axis. ˙ ψ→z ˙ φ→z We want ˙ ˙ ˙ ωx = θx + ψx + φx ˙ ˙ ˙ ωy = θy + ψy + φy ˙ ˙ ˙ ωz = θz + ψz + φz Lets start with ωz first to be different. ˙ z . So there is a component along z due to a changing ψ. we can see that θ is along the line of nodes. we just see that the angle between the line of nodes and the x axis is only φ. ˙ ˙ ˙ ˙ ˙ ωz = θz + ψz + φz = 0 + ψ cos θ + φ ˙ ˙ Now lets do the harder ones. that is θ revolves around the line of nodes. To find the x component of that. Add these all up for our total ωx .y = ψ sin θ.O. If we look at the diagram carefully on ˙ page 152. The adjacent side to φ with θ as the hypotenuse. ˙ ˙ φ it is in a whole different plane than x. even though the process is exactly the same. Look for ψy . For φy we note that φ revolves around the z axis. Thats good! So lets make it if ˙ φ = 0 we have the full ψ sin θ. Therefore no component in ˙ the y direction. so two projections are necessary to find its component. (throw ˙ ˙ ˙ in a negative). So we have ψy = −ψ sin θ cos φ. Project down to the xy ˙ ˙ plane like we did before. But we also have projected it in the opposite direction of the positive y direction. completely perpendicular to y. (ie multiply by cos φ because cos 0 = 1). φy = 0. θy = θ sin φ.y = ψ sin θ and now we remember that if φ = 0 we would have exactly placed it on top of the y axis. Its in a different plane again.˙ ˙ ˙ project it to the y axis. ψx. Add them all up ˙ ˙ ˙ ˙ ˙ ωy = θy + ψy + φy = θ sin φ − ψ sin θ cos φ + 0 Here is all the ω's together ˙ ˙ ωx = θ cos φ + ψ sin θ sin φ ˙ ˙ ωy = θ sin φ − ψ sin θ cos φ ˙ ˙ ωz = ψ cos θ + φ 8 . We are to find ψ = ω cos θt We know ω is directed north along the axis of rotation. θ.23. If we look at the components of ω, lets see where ω is.(θ is zero at the north pole. and call z the vertical direction pointed toward the sky.15. ω is aligned with y .13. 5. if we are at the equator.Homework 7: # 4. 5. Place ourselves in the coordinate system of whoever may be firing the projectile on the surface of the Earth. λ = π/2 − θ.22. we can take a hint from Goldstein's Figure 4. when ω and z are aligned). Foucault pendulum Michael Good Oct 9. If we are at the north pole. Note that the angle between z and ω is the co-latitude. the direction of deviation being to the right in the northern hemisphere. 5. Answer: I'll call the angular deviation ψ. that is. the angle from the poles to the point located on the surface of the Earth.21. sticking out of the north pole of the earth. With our coordinate system in hand. Show that to a first approximation the angular deviation from the direction of fire resulting from the Coriolis effect varies linearly with time at a rate ω cos θ where ω is the angular frequency of Earth's rotation and θ is the co-latitude. but horizontally north). Only ωz is used for our approximation. Parallel transport it to the surface and note that it is between y and z . We know θ is the co-latitude. It is clear that there is 1 . call x the horizontal direction pointed east. 2004 4. it is completely aligned with z . λ is the angle from the equator to the point located on the surface of the Earth. The latitude. that deflection of the horizontal trajectory in the northern hemisphere will depend on only the z component of ω. Call y the horizontal direction pointing north (not toward the north pole or into the ground. labeled ωz .22 A projectile is fired horizontally along Earth's surface. So following Goldstein's figure. Note that there is no Coriolis effect at the equator when θ = π/2. for ψ we can draw a triangle and note that the distance traveled by the projectile is just x = vt. (as explained on wolfram research) we can calculate the moment of inertia for the triangle.no component of ω in the x direction.15 Find the principal moments of inertia about the center of mass of a flat rigid body in the shape of a 45o right triangle with uniform mass density. with it situated with long side on the x-axis. because the Coriolis effect is Fc = −2m(ω × v) and ωy ×v would add a contribution in the z direction because our projectile is fired only along x and y . horizontally. 2 . The off-diagonal elements of the inertia tensor vanish. If we took into account the component in the y direction we would have an effect causing the particle to move into the vertical direction. d= xψ = d → ψ= d x vω cos θt2 = ω cos θt vt Therefore the angular deviation varies linearly on time with a rate of ω cos θ. 5. What are the principal axes? ψ= Answer: Using the moment of inertia formula for a lamina. 1 2 ac t = vω cos θt2 2 And using a small angle of deviation. we shall only be concerned with ωz . that is. therefore no angular deviation. while the y-axis cuts through the middle. which is a flat closed surface. Our acceleration due to the Coriolis force is ac = −2(ω × v) = 2(v × ω) The component of ω in the z direction is ωz = ω cos θ. Thus the magnitude of the acceleration is ac = 2vω cos θ The distance affected by this acceleration can be found through the equation of motion. you won't do the integral over again. 0). a . with r0 = a/3 2 IX = Ix − M r0 IY = Iy 2 IZ = Iz − M r0 These are 1 1 3 2 M a2 IX = (−)M a2 = (−)M a2 = 6 9 18 18 18 IY = M a2 6 3 . The center of mass is ycm = 2 ycm = 1 a2 a σ M a 0 0 a−x ydxdy = 2 a2 a 0 (a − x)2 dx 2 x3 3 a 0 (a2 − 2xa + x2)dx = a2 x − ax2 + 0 1 a = 2 a 3 From symmetry we can tell that the center of mass is (0.a a−x 0 Ix = σy 2 dxdy = 2 0 M 2 2M y dydx = 2 A a a 0 (a − x)3 dx 3 Solving the algebra. so if you're clever. Using the 3 parallel axis theorem. 2M 3a2 a Ix = (−x3 + 3ax2 − 3a2 x + a3)dx = 0 2M a2 8 1 6 M a2 [− −]= 3 4 4 4 6 For Iy a a−y 0 Iy = σx2 dxdy = 2 0 M 2 x dxdy A This has the exact same form. Iy = For Iz Iz = 1 1 M a2 σ(x2 + y 2)dxdy = Ix + Iy = (+)M a2 = 6 6 3 M a2 6 We can use the parallel axis theorem to find the principal moments of inertia about the center of mass. we may equate the torque to the moment of inertia times the angular acceleration.1 1 2 IZ = (−)M a2 = M a2 3 9 9 5. The equation of motion becomes 2 ¨ −lM g sin θ = (M rg + M l2)θ Using small oscillations. T . Compute the period for small oscillations in terms of the radius of gyration about the center of gravity and the separation of the point of suspension from the center of gravity. ¨ lF = I θ The force is −M g sin θ. then the sum of these distances is equal to the length of the equivalent simple pendulum. Then we get 4 . each having the same period.21 A compound pendulum consists of a rigid body in the shape of a lamina suspended in the vertical plane at a point other than the center of gravity. we can apply the small angle approximation sin θ ≈ θ 2 ¨ −lgθ = (rg + l2)θ lg ¨ θ+θ =0 2 rg + l 2 This is with angular frequency and period ω= lg + l2 → T = 2π = 2π ω 2 rg + l 2 lg 2 rg This is the same as the period for a physical pendulum T = 2π I = 2π M gl 2 rg + l 2 lg If we have two points of suspension. Show that if the pendulum has the same period for two points of suspension at unequal distances from the center of gravity. and l is the distance between the pivot point and center of gravity. and the moment of inertia. using the parallel axis theorem is 2 I = M rg + M l 2 where rg radius of gyration about the center of gravity. l1 and l2 . Answer: Looking for an equation of motion. 04 s. the door will slam shut as the automobile picks up speed. 2 2 rg rg + l1 + l 1 = + l2 + l1 l1 l2 2 rg (l2 − l1) + 2l1 = l2 + l1 l1 l2 This is only true if 2 rg = l 1 l 2 Thus our period becomes T = 2π 2 2 rg + l 1 = 2π l1 g 2 l 1 l 2 + l1 = 2π l1 g l2 + l 1 = 2π g L g where L is the length of a simple pendulum equivalent. So we get 2¨ mr0 θ = −amf sin θ Our equation of motion is 5 . the radius of gyration of the door about the axis of rotation is r0 and the center of mass is at a distance a from the hinges. Obtain a formula for the time needed for the door to close if the acceleration f is constant. because we are looking for l1 + l2 to be equivalent to a simple pendulum length. Answer: Begin by setting the torque equal to the product of the moment of inertia and angular acceleration. The force is F = −mf sin θ. If the hinges of the door are toward the front of the car. Show that if f is 0.2m wide. 5. the time will be approximately 3.2π This is 2 2 rg + l 1 = 2π l1 g 2 2 rg + l 2 l2 g 2 2 2 2 rg + l 1 rg + l 2 = l1 l2 And in a more favorable form.3m/s2 and the door is a uniform rectangle is 1.23 An automobile is started from rest with one of its doors initially at right angles. add l1 to both sides. ¨ I θ = aF 2 The moment of inertia is I = mr0 . while the math runs the show. and may be integrated.af ¨ θ = − 2 sin θ r0 This is rough. In our case we can not use the small angle approximation.wolfram. Here is a handy trick.com/EllipticIntegralSingularValue. 6 . 2 r0 2af π 2 T = √ 0 dθ = cos θ 2 r0 2af π 2 0 dθ = cos θ − cos π 2 2 r0 √ π 2K(sin) 2af 4 This can be seen from mathworld's treatment of elliptic integrals. denoted K. Now we have T = √ √ 2 r0 2 K( ) af 2 K(22) belongs to a group of functions called 'elliptic integral singular values'. π 2 T = 0 dt dθ = dθ π 2 0 dθ = ˙ θ π 2 0 2 r0 dθ √ 2af cos θ Here is where the physics takes a backseat for a few. ˙ ˙ ˙ dθ dθ dθ dθ ˙ d dθ ¨ = = = θ θ= dt dt dt dθ dt dθ Plug this into our equation of motion ˙ dθ ˙ af θ = − 2 sin θ dθ r0 This is separable. ˙ θ2 af = 2 cos θ 2 r0 ˙ θ= 2af 2 cos θ r0 The time may be found by integrating over the time of travel it takes for the door to shut.html.wolfram.com/EllipticIntegraloftheFirstKind. K(kr) A treatment of them and a table of their values that correspond to gamma functions are given here: . The door starts at 90o ! How do we go about solving this then? Lets try integrating it once and see how far we can get. If we throw in a − cos 90o we might notice that this integral is an elliptic integral of the first kind. at . com/EllipticLambdaFunction. half of the length of the car door. neglect change in height.efunda.04 s 3(.3m/s2 we have 2 r0 = T = 4a 1 √ (3. that is. I now have 1 Γ( ) = 3. demonstrating the Earth's rotation. assuming its mass is uniform. solve for ξ = x + iy Answer: The Foucault pendulum is a swinging weight supported by a long wire. K(k1) = Our time is now T = 1 2 r0 Γ2 (4) √ af 4 π Γ2 (1) √4 4 π Fortunately. Hint: neglect centrifugal force.14 Foucault Pendulum Find the period of rotation as a function of latitude.com/math/gamma/findgamma.3) 4 3.63)2 = 3. 2 I = M r0 = M a2 + M 2 4 a = M a2 3 3 we now have 4 2 a 3 With a = .wolfram.cfm. √ Our kr value of 22 corresponds to k1 . Move the axis to the edge of the rectangle using 3 the parallel axis theorem.63)2 = 3f 4 π 4(. I used this one . so that the wire's upper support restrains the wire only in the vertical direction and the weight is set swinging with no lateral or circular motion. there are nice calculators that will compute gamma functions quickly.6m. The plane of the pendulum gradually rotates.The 'elliptic lambda function' determines the value of kr .6) 1 √ (3. The moment of inertia of a uniform rectangle about the axis that bisects it is M a2 . A table of lambda functions is here . Solve 7 .63 4 Back to the physics. And with f = .035 ≈ 3. From the singular value table. that is. the acceleration from the tension and the Coriolis acceleration. It's solution is. y facing north. I have x facing east. we are concerned only with the x and y accelerations. the over damped case √g √g ξ = e−iω sin λt (Aei l t + Be−i l t) The equation for oscillation of a pendulum is g q+ q=0 ¨ l It has solution 8 g l >> . and z facing to the sky. using ω sin λ. The Coriolis acceleration is quickly derived ac = yω sin λˆ − xω sin λˆ + xω cos λˆ ˙ x ˙ y ˙ z Looking for the period of rotation. T − 2ω × vr m In my system. Our overall acceleration equations become g ˙ x = − x + 2yω sin λ ¨ l g y = − y − 2xω sin λ ¨ ˙ l The g/l terms were found using approximations for the tension components. The equation of motion for acceleration takes into account the vertical acceleration due to gravity. This yeilds ar = g + ωx = 0 ωy = ω sin θ = ω cos λ ωz = ω cos θ = ω sin λ The only velocity contributions come from the x and y components.for the period of rotation of this plane. Tx = −T x → Tx /ml = g/l and the same for y. for we can ignore the change in height. l Introducing ξ = x + iy and adding the two equations after multiplying the second one by i ¨ g ˙ ˙ ξ + ξ = −2ω sin λ(−y + ix) l ¨ g ˙ ξ + ξ = −2iω sin λξ l ¨ g ˙ ξ + ξ + 2iω sin λξ = 0 l This is the damped oscillation expression. and θ is the co-latitude. 2π 2π TEarth cos θ = → TF oucault = Tearth TF oucault cos θ This can be checked because we know the pendulum rotates completely in 1 day at the North pole where θ = 0 and has no rotation at the equator where θ = 900 . or ω sin λ where λ is the latitude.q = Aei √g l t + Be−i √g l t We can simplify our expression then. ω = 2π/T . The period can be found using. Chapel Hill has a latitude of 36o . a Foucault pendulum takes TF oucault = to make a full revolution. using q ξ = qe−iω sin λt Where the angular frequency of the plane's rotation is ω cos θ. 24 hours ≈ 41 hours sin 36o 9 . Homework 8: # 5. 5. (1. 5. for the generalized coordinate ψ. (5.4 Derive Euler's equations of motion. 5. from the Lagrange equation of motion.26 Michael Good Oct 21. in the form of Eq.39'). 2004 5.4. Eq.6. Answer: Euler's equations of motion for a rigid body are: I1 ω1 − ω2 ω3 (I2 − I3) = N1 ˙ I2 ω2 − ω3 ω1 (I3 − I1) = N2 ˙ I3 ω3 − ω1 ω2 (I1 − I2) = N3 ˙ The Lagrangian equation of motion is in the form ∂T d ∂T ( )− = Qj dt ∂ qj ˙ ∂qj The kinetic energy for rotational motion is 3 T = i 1 2 Ii ωi 2 The components of the angular velocity in terms of Euler angles for the body set of axes are ˙ ˙ ω1 = φ sin θ sin ψ + θ cos ψ ˙ ˙ ω2 = φ sin θ cos ψ − θ sin ψ ˙ ˙ ω3 = φ cos θ + ψ Solving for the equation of motion using the generalized coordinate ψ: d ∂T ∂T ( )− = Nψ ˙ dt ∂ ψ ∂ψ 3 Ii ωi i ∂ωi d − ∂ψ dt 1 3 Ii ωi i ∂ωi = Nψ ˙ ∂ψ .53).7. pulling out the negative sign on the second term. explicitly 3 Ii ωi i ∂ωi d − ∂ψ dt 3 Ii ωi i ∂ωi = Nψ ˙ ∂ψ d I3 ω3 = Nψ dt ˙ ˙ ˙ ˙ I1 ω1 (−θ sin ψ + φ sin θ cos ψ) + I2 ω2 (−θ cos ψ − φ sin θ sin ψ) − This is.Now is a good time to pause and calculate the partials of the angular velocities. I1 ω1 (ω2) − I2 ω2 (ω1) − I3 ω3 = Nψ ˙ I3 ω3 − ω1 ω2 (I1 − I2) = Nψ ˙ And through cyclic permutations I2 ω2 − ω3 ω1 (I3 − I1) = N2 ˙ I1 ω1 − ω2 ω3 (I2 − I3) = N1 ˙ we have the rest of Euler's equations of motion for a rigid body. ∂ω1 ˙ ˙ = −θ sin ψ + φ sin θ cos ψ ∂ψ ∂ω2 ˙ ˙ = −θ cos ψ − φ sin θ sin ψ ∂ψ ∂ω3 =0 ∂ψ and ∂ω2 ∂ω1 = =0 ˙ ˙ ∂ψ ∂ψ ∂ω3 =1 ˙ ∂ψ Now we have all the pieces of the puzzle. 2 . 6. and by already making the substitution. rigid body motion.5. Answer: Marion shows that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency ω more explicitly than Goldstein. rolling on a fixed cone in space whose axis is along the angular momentum. The angular velocity vector is along the line of contact of the two cones. ω1 = −(˙ 3 . • Show from parts (a) and (b) that the motion of the force-free symmetrical top can be described in terms of the rotation of a cone fixed in the body whose axis is the symmetry axis.5 cm apart on Earth's surface. Show that the same description follows immediately from the Poinsot construction in terms of the inertia ellipsoid. Show also that the symmetry axis rotates in space about the fixed direction of the angular momentum with angular frequency I3 ω3 ˙ φ= I1 cos θ where φ is the Euler angle of the line of nodes with respect to the angular momentum as the space z axis. The other Euler equations are I3 − I ω3)ω2 I I3 − I ω2 = −(˙ ω3)ω1 I Solving these. symmetric. show therefore that Earth's rotation axis and axis of angular momentum are never more than 1. we see that ω3 = constant. show that ω rotates in space ˙ about the angular momentum with the same frequency φ. but that the angle θ between ω and L is given by sin θ = Ω sin θ ˙ φ where θ is the inclination of ω to the symmetry axis. because we are dealing with constants. Chapter 4. Beginning with Euler's equation for force-free. • Using the results of Exercise 15. Using the data given in Section 5.6 • Show that the angular momentum of the torque-free symmetrical top rotates in the body coordinates about the symmetry axis with an angular frequency ω. Ω= we get I3 − I ω3 I (ω1 + iω2) − iΩ(ω1 + iω2) = 0 ˙ ˙ Let q = ω1 + iω2 Now q − iΩq = 0 ˙ has solution q(t) = AeiΩt this is ω1 + iω2 = A cos Ωt + iA sin Ωt and we see ω1 (t) = A cos Ωt ω2 (t) = A sin Ωt The x3 axis is the symmetry axis of the body. I3 ω3 ˙ φ= I1 cos θ To prove I3 ω3 ˙ φ= I1 cos θ We may look at the two cone figure angular momentum components. L1 = 0 L2 = L sin θ L3 = L cos θ If α is the angle between ω and the vertical body axis. then I3 − I ω3 I 4 . so the angular velocity vector precesses about the body x3 axis with a constant angular frequency Ω= . where L is directed along the vertical space axis and θ is the angle between the space and body vertical axis. For ω × x3 we have |ω × x3 | = ω sin θ = 2 2 ωx + ωy Using the angular velocity components in terms of Euler angles in the body fixed frame. this is equal to ˙ ωL sin θ = Lφ sin θ ˙ with θ fixed. (using the instant in time where x2 is in the plane of x3 . ω. and θ = 0. we may find. where ψ = 0) . 2 2 |ω × L| = ωL sin θ = L ωx + ωy Using the angular velocity components in terms of Euler angles in the space fixed frame. this is equal to 5 .ω1 = 0 ω2 = ω sin α ω3 = ω cos α The angular momentum components in terms of α may be found L1 = I1 ω1 = 0 L2 = I1 ω2 = I1 ω sin α L3 = I3 ω3 = I3 ω cos α Using the Euler angles in the body frame. and L. ˙ ˙ ω2 = φ sin θ cos ψ − θ sin ψ ˙ ω2 = φ sin θ This is ω2 ω sin α L2 L L ˙ φ= = = ω( ) = sin θ sin θ I1 ω L2 I1 Plugging in L3 L L3 I3 ω3 ˙ φ= = = I1 I1 cos θ I1 cos θ A simple way to show sin θ = Ω sin θ ˙ φ may be constructed by using the cross product of ω × L and ω × x3 . 5cm apart on the Earth's surface. cos θ ≈ 1. d= I3 − I1 s = (.˙ ω sin θ = φ sin θ Using these two expressions. and s is the average distance of separation.6 cm I1 Force free motion means the angular momentum vector L is constant in time and stationary. 5 m. (because the center of mass of the body is fixed). I3 > I1 and the data says there is 10m for amplitude of separation of pole from rotation axis. ω precesses 2 6 . Earth is considered an oblate spheroid. sin θ ≈ θ . and I1 /I3 ≈ 1.00327)(5) = 1. So because T = 1 ω · L is constant. Using sin θ = Ω sin θ ˙ φ I3 ω3 ˙ φ= I1 cos θ Ω= we have sin θ = I3 − I1 I1 cos θ ω3 sin θ I1 I3 ω3 I3 − I1 θ I1 I3 − I1 ω3 I1 Applying the approximations θ = θ = d I3 − I1 s = R I1 R where R is the radius of the Earth. as well as the rotational kinetic energy. sin θ ≈ θ . we may find their ratio ˙ ωL sin θ Lφ sin θ = ˙ ω sin θ φ sin θ ˙ sin θ ψ = ˙ sin θ φ ˙ Because ψ = Ω sin θ = Ω sin θ ˙ φ To show that the Earth's rotation axis and axis of angular momentum are never more than 1. which we will assume is half the amplitude. the following approximations may be made. This tracing is called the space cone. The Euler equations I1 ω1 − ω2 ω3 (I2 − I3) = 0 ˙ I2 ω2 − ω3 ω1 (I3 − I1) = 0 ˙ I3 ω3 − ω1 ω2 (I1 − I2) = 0 ˙ become 7 . So we have two cones. First lets define our object to have distinct principal moments of inertia. the body cone. Now the symmetry axis of the body has the angular velocity ω precessing around it with a constant angular frequency Ω. hugging each other with ω in the direction of the line of contact. Discuss the boundedness of the resultant motion for each of the three principal axes.7 For the general asymmetrical rigid body. This is because k and p are so small. while the product of components perpendicular to the axis can be neglected. The direction of ω is assumed to differ so slightly from a principal axis that the component of ω along the axis can be taken as constant. 204 by examining the solution of Euler's equations for small deviations from rotation about each of the principal axes. This results from I1 = I2 as shown below: L · (ω × e3) = 0 because ω × e3 = ω2 e1 − ω1 e2 L · (ω × e3) = I1 ω1 ω2 − I2 ω1 ω2 = 0 Because I1 = I2 . We have ω = ω1 e1 if we spin it around the x1 axis.around with a constant angle. I1 < I2 < I3 . Answer: Marion and Thornton give a clear analysis of the stability of a general rigid body. Apply some small perturbation and we get ω = ω1 e1 + ke2 + pe3 In the problem. verify analytically the stability theorem shown geometrically above on p. Lets examine the x1 axis first. Thus another cone is traced out. only if L is lined up with x3 space axis. x3 and ω all lie in the same plane will show that this space cone is traced out by ω. we are told to neglect the product of components perpendicular to the axis of rotation. 5. Proving that L. Solving the other two yields I3 − I1 ˙ k=(ω1)p I2 p=(˙ I1 − I2 ω1)k I3 To solve we may differentiate the first equation. Ω2 is imaginary and the perturbation increases forever with time. and the intermediate principal axis of rotation is unstable. Around the x2 axis we have unbounded motion. 8 . Thus we conclude that only the largest and smallest moment of inertia rotations are stable. we see ω1 is constant from the first equation. because I2 < I3 and we obtain a negative sign under the square root.I1 ω1 − kp(I2 − I3) = 0 ˙ I2 k − pω1 (I3 − I1) = 0 I3 p − ω1 k(I1 − I2) = 0 ˙ Neglecting the product pk ≈ 0. and plug into the second: I3 − I1 ¨ ω1)p ˙ k=(I2 Solve for k(t): k(t) = AeiΩ1k t + Be−iΩ1k t with Ω1k = ω1 Do this for p(t) and you get Ω1k = Ω1p ≡ Ω1 Cyclic permutation for the other axes yields Ω1 = ω1 (I1 − I3)(I1 − I2) I2 I3 (I2 − I1)(I2 − I3) I3 I1 (I3 − I2)(I3 − I1) I1 I2 (I1 − I3)(I1 − I2) I2 I3 → (I1 − I3)(I1 − I2) 2 ¨ k+(ω1)k = 0 I2 I3 Ω2 = ω 2 Ω3 = ω 3 Note that the only unstable motion is about the x2 axis. 26 For the axially symmetric body precessing uniformly in the absence of torques.47) of Goldstein. ˙ ˙ ω1 = φ sin θ sin ψ + θ cos ψ ˙ ˙ ω2 = φ sin θ cos ψ − θ sin ψ ˙ ˙ ω3 = φ cos θ + ψ we have ˙ ω1 = φ sin θ sin ψ + θ cos ψ = A sin(Ωt + δ) ˙ ˙ ω2 = φ sin θ cos ψ − θ sin ψ = A cos(Ωt + δ) ˙ ˙ ω3 = φ cos θ + ψ = constant (1) (2) (3) Multiplying the left hand side of (1) by cos ψ and the left hand side of (2) by sin ψ. we have I1 = I2 . symmetry axis Lz . and Euler's equations are I1 ω1 = (I1 − I3)ω2 ω3 ˙ I2 ω2 = (I3 − I1)ω1 ω3 ˙ I3 ω3 = 0 ˙ This is equation (5. only without the typos. ω1 = A cos Ωt ω2 = A sin Ωt where Ω= I3 − I1 ω3 I1 Using the Euler angles in the body fixed frame. Following Goldstein.5. Answer: For an axially symmetric body. and subtracting them yields ˙ ˙ ˙ ˙ [φ sin θ sin ψ cos ψ + θ cos2 ψ] − [φ sin θ cos ψ sin ψ − θ sin2 ψ] = θ 9 . find the analytical solutions for the Euler angles as a function of time. From this. Thus Ωt + δ + ψ = nπ with n = 0. ψ = −Ω.Thus we have ˙ θ = A sin(Ωt + δ) cos ψ + A cos(Ωt + δ) sin ψ ˙ θ = A sin(Ωt + δ + ψ) ˙ I assume uniform precession means θ = 0.. if n = 0 ψ = −Ωt + ψ0 ˙ where ψ0 is the initial angle from the x − axis. and add them: ˙ ˙ ˙ ˙ [φ sin θ sin2 ψ + θ cos ψ sin ψ] + [φ sin θ cos2 ψ − θ sin ψ cos ψ] = φ sin θ Thus we have ˙ φ sin θ = A sin(Ωt + δ) sin ψ + A cos(Ωt + δ) cos ψ ˙ φ sin θ = A cos(Ωt + δ + ψ) Plugging this result into (3) ω3 = A cos θ ˙ cos(Ωt + ψ + δ) + ψ sin θ ˙ Using ψ = −Ω and Ωt + δ + ψ = 0. ω3 = A cos θ cos(0) − Ω sin θ A = (ω3 + Ω) tan θ and since Ω = I3 −I1 I1 ω 3 A = (ω3 + I3 − I1 I3 ω3) tan θ = ω3 tan θ I1 I1 With this we can solve for the last Euler angle. cos(Ωt + ψ + δ) I3 cos(0) ˙ φ=A = ω3 tan θ sin θ I1 sin θ 10 . φ. ±2. ±1.. no nutation or bobbing up and down.. If we multiply the left hand side of (1) by sin ψ and the left hand side of (2) by cos ψ. I3 ω3 ˙ φ= I1 cos θ φ= So all together θ = θ0 ψ(t) = −Ωt + ψ0 φ(t) = I3 ω3 t + φ0 I1 cos θ I3 ω3 t + φ0 I1 cos θ 11 . 24. Answer: Let x = x + l sin θ z = ax2 − l cos θ Then T = 1 m(x 2 + z 2) ˙ ˙ 2 U = mgz Solving in terms of generalized coordinates.19.25 Michael Good Nov 2. our Lagrangian is 1 ˙ ˙ m(x2 +2xl cos θθ+4a2 x2 x2 +4axxlθ sin θ+l2 θ2)−mg(ax2 −l cos θ) ˙ ˙ ˙ ˙ 2 L = T −U = Using 1 qT ˙ ˙ L = L0 + ˜ q 2 where q and T are matrices.19 The point of suspension of a simple pendulum of length l and mass m is constrained to move on a parabola z = ax2 in the vertical plane. Obtain the Hamilton's equations of motion. 8. Derive a Hamiltonian governing the motion of the pendulum and its point of suspension. We can see ˙ q= ˙ T = with 1 x ˙ ˙ θ m(1 + 4a2 x2) ml(cos θ + 2ax sin θ) ml(cos θ + 2ax sin θ) ml2 . 8.Homework 9: # 8. x and θ. 2004 8. Y.L0 = −mg(ax2 − l cos θ) The Hamiltonian is H= Inverting T by a b c d with the algebra. = So now we have 1 m2 l2 Y 1 mY ml2 −ml(cos θ + 2ax sin θ) −ml(cos θ + 2ax sin θ) m(1 + 4a2 x2) 1 −(cos θ + 2ax sin θ)/l −(cos θ + 2ax sin θ)/l (1 + 4a2 x2)/l2 m2 l2 (sin θ 1 1 ≡ 22 2 − 2ax cos θ) m l Y T −1 = T −1 = I want to introduce a new friend. lets call him J J ≡ (cos θ + 2ax sin θ) Y ≡ (sin θ − 2ax cos θ)2 So. 1 1 = 22 ad − bc m l (1 + 4ax2) − m2 l2 (cos θ + 2ax sin θ)2 this is = m2 l2 (sin2 = θ+ 4ax2 1 − 4ax cos θ sin θ − 4a2 x2 sin2 θ) −1 1 −1 pT p − L0 ˜ 2 = 1 ad − bc d −b −c a m2 l2 (sin2 1 θ − 4ax sin θ cos θ + 4a2 x2 cos2 θ) which I'll introduce. H= 1 −1 pT p − L0 ˜ 2 2 . for simplicity's sake. T −1 = 1 mY 1 −J/l −J/l (1 + 4a2 x2)/l2 Proceed to derive the Hamiltonian. and patiently. I then broke each p term and began grouping them.we can go step by step. They are x= ˙ ∂H ∂px ∂H ˙ θ= ∂pθ px = − ˙ ∂H ∂x pθ = − ˙ ∂H ∂θ The first two are easy. Go slowly. 1 J 1 cos θ + 2ax sin θ [px − pθ] = [px − pθ] mY l m(sin θ − 2ax cos θ)2 l x= ˙ ˙ θ= 1 1 + 4a2 x2 1 1 + 4a2 x2 [−Jpx + pθ] = [−(cos θ+2ax sin θ)px + pθ] 2 mY l l ml(sin θ − 2ax cos θ) l But the next two are far more involved. I handled the partial with respect to x by taking the product rule between the two main pieces. especially with my substitutions. After taking the derivative before grouping. 1 mY 1 −J/l −J/l (1 + 4a2 x2 /l2 px pθ 1 mY px − (J/l)pθ (−J/l)px + (1 + 4a2 x2 /l2)pθ T −1 p = and = J J 1 + 4a2 x2 2 1 pθ) (p2 − pθ px − pθ px + x mY l l l2 the full Hamiltonian is pT −1 p = ˜ 1 J 1 + 4a2 x2 2 (p2 − 2 pθ px + pθ) + mg(ax2 − l cos θ) x 2mY l l2 plugging in my Y and J H= 1 cos θ + 2ax sin θ 1 + 4a2 x2 2 (p2 − 2 pθ p x + pθ)+mg(ax2 −l cos θ) 2m(sin θ − 2ax cos θ)2 x l l2 H= Now to find the equations of motion. my px looked like this: ˙ px = − ˙ ∂H ∂x ∂H 1 −4a sin θ 8a2 x = [p θ px + 2 p2] θ 2 ∂x 2m(sin θ − 2ax cos θ) l l − −2(−2a cos θ) cos θ + 2ax sin θ 1 + 4a2 x2 [p2 − 2 pθ px + pθ] + 2mgax 2m(sin θ − 2ax cos θ)3 x l l2 3 . and mess inside the parenthesis that has p terms. the fraction out front. of course pθ = − ˙ 1 2 sin θ 4ax cos θ ∂H = [−]pθ px 2 ∂θ 2m[sin θ − 2ax cos θ] l l +[p2 − 2 x cos θ + 2ax sin θ 1 + 4a2 x2 2 −2(cos θ + 2ax sin θ) pθ p x + pθ][+ mgl sin θ l l2 2m[sin θ − 2ax cos θ]3 − separating terms. you get a monster. −(cos θ + 2ax sin θ) 2 p m[sin θ − 2ax cos θ]3 x cos θ + 2ax sin θ 1 + 4a2 x2 2 pθ m[sin θ − 2ax cos θ]3 l2 and the longest one. pθ ˙ ∂H 1 1 + 4a2 x2 2 = [(cos θ + 2ax sin θ)(p2 + pθ) x ∂θ m[sin θ − 2ax cos θ]3 l2 − [(sin θ − 2ax cos θ)2 + 2(cos θ + 2ax sin θ)2] pθ px l 4 − .. [(sin θ − 2ax cos θ)2 2(cos θ + 2ax sin θ)2 +]pθ px lm[sin θ − 2ax cos θ]3 lm[sin θ − 2ax cos θ]3 add them all up for the fourth equation of motion.. 4a(cos θ + 2ax sin θ) 2 p 2ml2 [sin θ − 2ax cos θ]3 θ 4a cos θ p2 2m[sin θ − 2ax cos θ]3 x and the longest one.. for px : ˙ 2a cos θ + 2ax sin θ 2 2 − sin2 θ + 2ax sin θ cos θ ∂H =− [cos θp2 + pθ − px pθ]−2mgax x ∂x m[sin θ − 2ax cos θ]3 l2 l Now for the next one. 2a [sin2 θ − 2 − 2ax cos θ sin θ]pθ px lm[sin θ − 2ax cos θ]3 Adding them all up yields. Lets group the p terms.. pθ : ˙ ∂H ∂θ Taking the derivative.Now start simplifying. 24 A uniform cylinder of radius a and density ρ is mounted so as to rotate freely around a vertical axis. The hardest part of this Lagrangian to understand is likely the translational energy due to the particle. The moment of inertia of a cylinder is 1 1 M a2 = ρπha4 2 2 There are three forms of kinetic energy in the Lagrangian. and φ the rotational angle of the cylinder. the rotational energy of the particle. arrive at a Hamiltonian for the combined system of particle and cylinder. MathWorld gives a treatment of this one under helix.Together in all their glory: ∂H 1 1 + 4a2 x2 2 [(cos θ + 2ax sin θ)(p2 + pθ) = x ∂θ m[sin θ − 2ax cos θ]3 l2 − [(sin θ − 2ax cos θ)2 + 2(cos θ + 2ax sin θ)2] pθ px l pθ = − ˙ ˙ px = − ˙ cos θ + 2ax sin θ 2 2 − sin2 θ + 2ax sin θ cos θ 2a ∂H [cos θp2 + pθ − =− px pθ]−2mgax x 3 ∂x m[sin θ − 2ax cos θ] l2 l x= ˙ 1 cos θ + 2ax sin θ [px − pθ] m(sin θ − 2ax cos θ)2 l ˙ θ= 1 + 4a2 x2 1 [−(cos θ + 2ax sin θ)px + pθ] ml(sin θ − 2ax cos θ)2 l 8. The relationship between height and angle of rotational for a helix is I= h = cθ Where c is the distance between the coils of the helix. Answer: My generalized coordinates will be θ. Understand that if the cylinder was not rotating 5 . The rotational energy of the cylinder. The only potential energy of the system will be the potential energy due to the height of the particle. On the outside of the cylinder is a rigidly fixed uniform spiral or helical track along which a mass point m can slide without friction. the rotational angle of the particle with respect to the cylinder. Using any set of coordinates. and solve for the motion of the system. and the translational kinetic energy of the particle. Suppose a particle starts at rest at the top of the cylinder and slides down under the influence of gravity. ˙ then the rotational kinetic energy of the particle would merely be m a2 θ2 . we can solve for the motion of the system. (duh!) Here are the EOM: − ∂H = pθ = mgc ˙ ∂θ ∂H − = pφ = 0 ˙ ∂φ ∂H (I + ma2)pθ − ma2 pφ ˙ =θ= ∂pθ m(a2 + c2)(I + ma2) − m2 a4 ∂H −ma2 pθ + pφ m(a2 + c2) ˙ =φ= ∂pφ m(a2 + c2)(I + ma2) − m2 a4 ˙ To solve for the motion. T = ma2 + mc2 ma2 ma2 I + ma2 q= ˙ ˙ θ ˙ φ 1 ˙2 m 2 ˙ ˙2 I φ + [a (θ + φ) + c2 θ2] + mgcθ 2 2 Using the same 2 by 2 inverse matrix form from the previous problem. but 2 the rotation of the cylinder is adding an additional rotation to the particle's position. L= This is 1 L = L0 + ˜ T q q ˙ ˙ 2 Solve for T. lets use the boundary conditions. 1 + ma2) − m2 a4 I + ma2 −ma2 −ma2 m(a2 + c2) T −1 = (ma2 + mc2)(I Now we can find the Hamiltonian. θ(0) = φ(0) = 0 leads to pφ (0) = pθ (0) = 0 leads to pθ = mgct pφ = 0 6 . we may solve for T −1 . Lets write down the Lagrangian. H= This is H= p2 (I + ma2) − 2ma2 pθ pφ + p2 m(a2 + c2) θ φ − mgcθ 2[m(a2 + c2)(I + ma2) − m2 a4] 1 p T −1 p − L0 ˜ 2 From the equations of motion. φ = ωt. the particle moves through an angle ψ = θ + φ. we obtain φ= θ= −mgct2 1 2[mc2 + 2 M (a2 + c2)] 1 (m + 2 M)gct2 1 2[mc2 + 2 M (a2 + c2)] 8. Answer: In the laboratory system. Set up the Hamiltonian for the particle in an inertial system of coordinates and also in a system fixed in the rotating cylinder. ˙ ˙ Pluggin and chuggin into θ and φ and integrating. Identify the physical nature of the Hamiltonian in each case and indicate whether or not the Hamiltonians are conserved.25 Suppose that in the previous exercise the cylinder is constrained to rotate uniformly with angular frequency ω. The cylinder moves uniformly. yields the motion φ= −m2 a2 gct2 + c2)(I + ma2) − m2 a4] 2[m(a2 θ= (I + ma2)mgct2 2[m(a2 + c2)(I + ma2) − m2 a4] 1 If we plug in I = M a2 where M is the mass of the cylinder. so the kinetic energy T = may be expressed 1 1 ˙ ˙ ma2 ψ 2 + mc2 (ψ − ω)2 2 2 The potential energy may be written T = U = −mgc(ψ − ωt) So we have L= 1 ˙ ˙ m(a2 ψ 2 + c2 (ψ − ω)2) + mgc(ψ − ωt) 2 ∂L ˙ ˙ = p = ma2 ψ + mc2 (ψ − ω) ∂q ˙ and with H= 1 (˜ − a)T −1 (˜ − a) − L0 p p 2 7 1 1 ˙ ˙ ma2 (θ + φ)2 + mc2 θ2 2 2 . we find T −1 from 1 ˜ −1 q T q + qa + L0 ˙ ˙ ˙ 2 We can see things better if we spread out L L= L= so L0 = and T = [ma2 + mc2] T −1 = 1 m(a2 + c2) 1 2 2 mc ω + mgc(ψ − ωt) 2 1 1 ˙ ˙ 1 ma2 ψ 2 + mc2 ψ 2 − mc2 ω ψ + mc2 ω 2 + mgc(ψ − ωt) 2 2 2 Therefore. For the Hamiltonian in the rotating cylinder's frame. ψ = θ + φ = θ + ωt ˙ ˙ ˙ ψ =θ+φ=θ+ω T = 1 1 ˙ ˙ ma2 (θ + ω)2 + mc2 θ2 2 2 U = −mgcθ L= 1 1 ˙ ma2 (θ + ω)2 + mc2 θ2 + mgcθ 2 2 1 L = ˜ T q + qa + L0 q ˙ ˙ ˙ 2 Spread out L 1 1 ˙ ˙ [ma2 + mc2]θ2 + ma2 θω + ma2 ω2 + mgcθ 2 2 It becomes clear that L= 8 . therefore it is not the total energy. we have Hlab = (p − mc2 ω)2 mc2 ω 2 − − mgc(ψ − ωt) 2(a2 + c2) 2m(a 2 This is dependent on time. we express the movement in terms of the angle θ this is with respect to the cylinder. for our Hamiltonian. T = [ma2 + mc2] T −1 = L0 = Using again. 9 . thus conserved. it is time-independent. H= we may write H= 1 (p − ma2 ω)2 − ma2 ω − mgcθ 2 + c2) 2m(a 2 1 (p − a)T −1 (p − a) − L0 2 1 ma2 + mc2 1 ma2 ω + mgcθ 2 This is not explicitly dependent on time. 9. 9. Q = q cos α − p sin α P = q sin α + p cos α satisfies the symplectic condition for any value of the parameter α.2 Show that the transformation for a system of one degree of freedom.16.2.6. 9. What is the physical significance of the transformation for α = 0? For α = π/2? Does your generating function work for both of these cases? Answer: The symplectic condition is met if ˜ MJM = J We can find M from ˙ ζ = Mη ˙ which is ˙ Q ˙ P We know J to be J= ˜ Solving M J M we get ˜ M (J M) = M − sin α − cos α − sin α − cos α 1 cos α − sin α cos α − sin α 0 1 −1 0 0 −1 1 0 = cos α sin α − sin α cos α q ˙ p ˙ ˜ MJM = cos α sin α − sin α cos α = .31 Michael Good Nov 2.Homework 10: # 9. 2004 9. Find a generating function for the transformation. α = nπ. α = nπ. Rearranging to solve for p(Q. it along with its relevant equation is F1 = − P = q sin α − Q cos α q cos2 α + sin α sin α ∂F1 ∂Q ∂F1 ∂q q cos α Q + sin α sin α P =− Integrating −F1 = qQ sin α − 1 Q2 cot α + qQ(− sin α) + h(q) 2 sin α Q2 qQ cot α + + h(q) 2 sin α −F1 = − F1 = Using both F1 's we find Q2 qQ cot α − + h(q) 2 sin α Qq 1 + (q 2 + Q2) cot α sin α 2 This has a problem. and have it work for the holes. q) we have p=− The related equation for F1 is p= Integrating for F1 yields Qq q 2 cos α + + g(Q) sin α 2 sin α Solve the other one. and check at the end if there are problems with it. F2 (q. But otherwise its ok.Therefore ˜ MJM = J and the symplectic condition is met for this transformation. when α = nπ. lets put the condition. It blows up. If we solve for F2 we may be able to find out what the generating function is. I will first attempt an F1 type and proceed to solve. sky high. q). that is P (Q. t)'s relevant equations are F1 = − p= ∂F2 ∂q 2 . To find the generating function. P. or no rotation.p= F2 = and P q sin α − cos α cos α Pq q2 − tan α + f (P) cos α 2 ∂F2 ∂P Q = q cos α − (P − q sin α) tan α Q= F2 = qP cos α − F2 = qP (cos α + F2 = So therefore 1 qP F2 = − (q 2 + P 2) tan α + 2 cos α 1 This works for α = nπ but blows sky high for α = (n+ 2)π. So I'll put a con1 dition on F2 that α = (n + 2)π. The physical significance of this transformation for α = 0 is easy to see cause we get P2 sin2 α tan α + qP + g(q) 2 cos α 1 P2 − cos α) − tan α + g(q) cos α 2 P2 qP − tan α + g(q) cos α 2 Q = q cos 0 − p sin 0 = q P = q sin 0 − p cos 0 = p This is just the identity transformation. For α = π/2 we get π π − p sin = −p 2 2 π π P = q sin − p cos = q 2 2 Where the p's and q's have been exchanged. Q = q cos 3 . 9. • Show that the function that generates this transformation is F3 = −(eQ − 1)2 tan p Answer: Q and P are considered canonical variables if these transformation equations satisfy the symplectic condition. P are canonical variables if q and p are. ˜ MJM = J Finding M : ˙ ζ = Mη ˙ Q ˙ P Mij = ∂ζi ∂ηj q ˙ p ˙ ∂Q ∂q ∂P ∂q ∂Q ∂p ∂P ∂p =M M= q −1/2 cos p ∂Q = ∂q 2(1 + q 1/2 cos p) ∂Q −q 1/2 sin p = ∂p 1 + q 1/2 cos p ∂P = q −1/2 sin p + 2q cos p sin p ∂q ∂P = 2q 1/2 cos p + 2q cos2 p − 2q sin2 p ∂p Remembering cos2 A − sin2 A = cos 2A and 2 sin A cos A = sin 2A cos(A − B) = cos A cos B + sin A sin B 4 .6 The transformation equations between two sets of coordinates are Q = log(1 + q 1/2 cos p) P = 2(1 + q 1/2 cos p)q 1/2 sin p • Show directly from these transformation equations that Q. and we are left with some algebra for the off-diagonal terms. eh? Suddenly ugly became pretty. To show that F3 = −(eQ − 1)2 tan p 5 . The same works for messy except it becomes positive 1 because it has no negative terms out front. So finally we get ˜ M JM = 0 −1 1 0 =J which is the symplectic condition. ˜ M JM = Lets solve for ugly. which proves Q and P are canonical variables. q −1/2 cos p 2(1+q 1/2 cos p) −1/2 −q 1/2 sin p 1+q 1/2 cos p JM = 0 −1 1 0 q sin p + sin 2p 2q 1/2 cos p + 2q cos 2p JM = Now q −1/2 sin p + sin 2p 2q 1/2 cos p 2q −1/2 cos p − 2(1+q1/2 cos p) 2q 1/2 cos p + 2q cos 2p q 1/2 sin p 1+q 1/2 cos p ˜ M JM = q −1/2 cos p 2(1+q 1/2 cos p) −q 1/2 sin p 1+q 1/2 cos p q −1/2 sin p + sin 2p 2q 1/2 cos p + 2q cos 2p q −1/2 sin p + sin 2p q −1/2 cos p − 2(1+q1/2 cos p) 2q 1/2 cos p + 2q cos 2p q 1/2 sin p 1+q 1/2 cos p You may see that the diagonal terms disappear.we can proceed with ease. q −1/2 cos p −q 1/2 sin p (q −1/2 sin p+sin 2p)− (2q 1/2 cos p+2q cos 2p) 1 + q 1/2 cos p 2(1 + q 1/2 cos p) ugly = − sin2 p − q 1/2 sin p sin 2p − cos2 p − q 1/2 cos p cos 2p 1 + q 1/2 cos p ugly = −(1 + q 1/2 (cos 2p cos p + sin 2p sin p)) 1 + q 1/2 cos p ugly = −(1 + q 1/2 cos p) = −1 1 + q 1/2 cos p 0 ugly messy 0 ugly = Not so ugly anymore. obtain formulas for evaluating the Poisson brackets ˙ [φ. f] I1 sin2 θ 6 . and f is any arbitrary function of the Euler angles. Answer: Poisson brackets are defined by [u. we learned I1 b − I1 a cos θ pφ − pψ cos θ ˙ φ= I1 sin2 θ I1 sin2 θ So calculating ˙ [φ. v]q. 9. f (θ. ψ)] ˙ [ψ. and then solve for our transformation equations. φ. and ψ are the Euler angles.p = ∂u ∂v ∂u ∂v − ∂qi ∂pi ∂pi ∂qi From Goldstein's section on Euler angles. φ.generates this transformation we may take the relevant equations for F3 . f] = [pφ − pψ cos θ . P = 2(1 + q 1/2 cos p − 1) tan p(1 + q 1/2 cos p) P = 2q 1/2 sin p(1 + q 1/2 cos p) Thus F3 is the generating function of our transformation equations.16 For a symmetric rigid body. f (θ. φ. now lets plug this into the expression for P and put P in terms of q and p to get the other one. q=− P =− Solving for Q q = (eQ − 1)2 sec2 p √ 1+ q = eQ ∂F3 = −[−(eQ − 1)2 sec2 p] ∂p ∂F3 = −[−2(eQ − 1) tan p]eQ ∂Q sec2 p Q = ln(1 + q 1/2 cos p) This is one of our transformation equations. solve them. f] = For the next relation. ω = 7 . f] = 1 ∂f ∂f (I3 cos θ − (I3 cos2 θ + I1 sin2 θ)) ∂φ ∂ψ I3 I1 sin2 θ Both together. t) = ln(p + imωq) − iωt.Note that f = f (θ. f] = ∂f 1 ∂f 2 (− ∂ψ + ∂ψ cos θ) I1 sin θ 1 ∂f ∂f 2 2 2 (I3 cos θ ∂φ − (I3 cos θ + I1 sin θ) ∂ψ) I3 I1 sin θ 9. p. f] = −(1 cos2 θ ∂f cos θ ∂f) + −(−) + I3 I1 sin2 θ ∂ψ I1 sin2 θ ∂ψ ˙ ∂ ψ ∂f ∂pi ∂qi 1 cos θ ∂f ∂f)+( ) I1 sin2 θ ∂φ I1 sin2 θ ∂ψ 1 ∂f ∂f (− + cos θ) I1 sin2 θ ∂ψ ∂ψ I1 sin2 θ I3 cos θ ∂f I3 cos2 θ ∂f ˙ [ψ. f] = − ˙ ∂ φ ∂f ∂pi ∂qi ˙ Taking only two derivatives because φ doesn't depend on pθ . f] = ˙ [ψ. So our definition becomes ˙ [φ.31 Show by the use of Poisson brackets that for one-dimensional harmonic oscillator there is a constant of the motion u defined as k m What is the physical significance of this constant of motion? u(q. ψ) and not of momenta. f] = − and pψ pφ − pψ cos θ ˙ ψ= cos θ − I3 I1 sin2 θ This yields ˙ [ψ. ˙ [ψ. We get ˙ [φ. f] = (− ˙ [φ. f] = −(+ 2 2) ∂ψ + I3 I1 sin θ I3 I1 sin θ I3 I1 sin2 θ ∂φ ˙ [ψ. in final form ˙ [φ. φ. Answer: Goldstein verifies Jacobi's identity [u. u]] + [w. p) = So we have du ∂u ∂H ∂u ∂H ∂u = − + dt ∂q ∂p ∂p ∂q ∂t du imω p 1 = ( )− (kq) − iω dt p + imωq m p + imωq du iωp − kq iωp − mω 2 q = − iω = − iω dt p + imωq p + iωmq du = iω − iω = iω − iω dt p + imωq du =0 dt Its physical significance relates to phase. w]] + [v. Show [f. [v. The Hamiltonian is H(q. gh] = g[f. I will follow his lead. [u. H] + dt ∂t which we must prove equals zero if u is to be a constant of the motion. kq 2 p2 + 2m 2 Show Jacobi's Identity holds. v]] = 0 using an efficient notation.Answer: We have du ∂u = [u. h] + [f. v] = ui Jij vj 8 . If we say ui ≡ ∂u ∂ηi vij ≡ ∂v ∂ηi ∂ηj Then a simple way of expressing the Poisson bracket becomes apparent [u. [w. g]h where the brackets are Poisson. u]] + [w. [v. gh] = [f. Add them up. g]h + g[f. gh] = ∂f ∂g ∂f ∂h ∂f ∂h ∂f ∂g h− h+g −g ∂qi ∂pi ∂pi ∂qi ∂qi ∂pi ∂pi ∂qi [f. w]] + [v. Here we have [u.This notation becomes valuable when expressing the the double Poisson bracket. [u. u]]. w we see there are only two terms that show up Jij Jkl ui vk wlj and Jji Jkl ui vk wjl The first from [u. gh] = Grouping terms [f. remembering Jkl are just constants. [w. [w. w]] = ui Jij (vkj Jkl wl + vk Jkl wlj) doing this for the other two double Poisson brackets. [v. v]] = 0 Its ok to do the second property the long way: [f. gh] = [f. Looking at one double partial term. [v. w]j = ui Jij (vk Jkl wl)j Taking the partial with respect to ηj we use the product rule. realizing order of partial is immaterial. Therefore [u. w]] = ui Jij [v. [v. for a total of 6. h] ∂f ∂(gh) ∂f ∂(gh) − ∂qi ∂pi ∂pi ∂qi ∂f ∂g ∂h ∂f ∂h ∂g (h+g)− (g + h) ∂qi ∂pi ∂pi ∂pi ∂qi ∂qi 9 . we get 4 more terms. [u. w]] and the second from [v. and J is antisymmetric: (Jij + Jji)Jkl ui vk wlj = 0 All the other terms are made of second partials of u or v and disappear in the same manner. • Use the results above to reduce to quadratures the problem of point particle of mass m moving in the gravitational field of two unequal mass points fixed on the z axis a distance 2a apart. z. 2004 10.Homework 11: # 10. v. and means to just get the problem into a form where the only thing left to do is take an integral. φ by the equations.7 • A single particle moves in space under a conservative potential. 10. Set up the Hamilton-Jacobi equation in ellipsoidal coordinates u. φ) is the equation separable. This is an old usage of the word quadratures. Answer: Let's obtain the Hamilton Jacobi equation. This will be used to reduce the problem to quadratures. v.7 b. r = a sinh v sin u z = a cosh v cos u For what forms of V (u. 10.26 Michael Good Nov 2. φ defined in terms of the usual cylindrical coordinates r.17. Here T = 1 2 1 1 ˙ mr + mz 2 + mr2 φ2 ˙ ˙ 2 2 2 r = a sinh v sin u r = a cosh v sin uv + a sinh v cos uu ˙ ˙ ˙ z = a cosh v cos u z = a sinh v cos uv − a cosh v sin uu ˙ ˙ ˙ Here r2 + z 2 = a2 (cosh2 v sin2 u+sinh2 v cos2 u)(v 2 + u2) = a2 (sin2 u+sinh2 v)(v 2 + u2) ˙ ˙ ˙ ˙ ˙ ˙ 1 . φ) = (sin2 u+sinh2 v)E 2ma2 ∂u ∂v 2ma2 sinh2 v sin2 u φ A little bit more work is necessary.To express in terms of momenta use pv = pu = ∂L = ma2 (sin2 u + sinh2 v)v ˙ ∂v ˙ ∂L = ma2 (sin2 u + sinh2 v)u ˙ ∂u ˙ because the potential does not depend on v or u. I suggest drawing a picture. with the origin being between them. the principle function applies S = Wu + Wv + αφ φ − Et So our Hamilton Jacobi equation is 1 ∂Wu 2 ∂Wv 2 1 ∂Wφ 2 [( ) +( )]+ ( ) +V (u. at which point we will have only integrals to take. |r a˜|2 = (z z a)2 + r2 Using the results from part (a) for r and z. with two point masses on the z axis. |r a˜|2 = a2 (cosh v cos u z 2 1)2 + a2 sinh2 v sin2 u . so they are each a distance a from the origin. remembering we are in cylindrical coordinates. φ) we can then separate this equation into u. The potential is then formed from two pieces V =− GmM1 GmM2 − |r − a˜| |r + a˜| z z To solve for the denominators use the Pythagorean theorem. v. The cyclic coordinate φ ˙ ˙ yields a constant I'll call αφ ˙ pφ = mr2 φ = αφ So our Hamiltonian is H= p2 p2 + p 2 φ u v + +V 2 2 (sin2 u + sinh v) 2 sinh2 v sin2 u 2ma 2ma To find our Hamilton Jacobi expression. φ) = E 2 2 sinh2 v sin2 u ∂v ∂φ u + sinh v) ∂u 2ma 2ma2 (sin2 This is ∂Wu 2 ∂Wv 2 1 1 1 1 [( ) +( )]+ (+)α2 +(sin2 u+sinh2 v)V (u. Once we solve for V (u. v. v and φ parts. v. introducing another constant.|r a˜|2 = a2 (cosh2 v cos2 u z 2 cosh v cos u + 1 + sinh2 v sin2 u) Lets rearrange this to make it easy to see the next step. and go ahead and separate out u and v terms. A: 2 αφ 1 1 ∂Wu 2 1 − Gm(M1 − M2) cos u − E sin2 u = A ( ) + 2ma2 ∂u 2ma2 sin2 u a 2 αφ 1 ∂Wv 2 1 1 ( ) + − Gm(M1 − M2) cosh v − E sinh2 v = −A 2 2 2ma ∂v 2ma sinh2 v a The problem has been reduced to quadratures. |r a˜|2 = a2 (sinh2 v sin2 u + cosh2 v cos2 u + 1 z 2 cosh v cos u) Now convert the sin2 u = 1 − cos2 u and convert the cosh2 v = 1 + sinh2 v |r a˜|2 = a2 (sinh2 v + cos2 u + 1 z 2 2 cosh v cos u) Add the 1 and cosh v |r a˜|2 = a2 (cosh2 v + cos2 u z 2 |r So our potential is now V =− V =− GmM2 GmM1 − a(cosh v − cos u) a(cosh v + cos u) a˜|2 = (a(cosh v z 2 cosh v cos u) cos u))2 1 GmM1 (cosh v + cos u) + GmM2 (cosh v − cos u) a cosh2 v − cos2 u Note the very helpful substitution cosh2 v − cos2 u = sin2 u + sinh2 v Allowing us to write V V =− 1 GmM1 (cosh v + cos u) + GmM2 (cosh v − cos u) a sin2 u + sinh2 v Plug this into our Hamilton Jacobi equation. 3 . 10. and a cyclic coordinate has the characteristic component Wqi = qi αi . αx . y. y. Answer: I'm going to assume the angle is θ because there are too many α's in the problem to begin with. y. α. we set up the Hamiltonian-Jacobi equation by setting p = ∂S/∂q and we get H= 1 ∂S 2 1 ∂S 2 ∂S ( ) + ( ) + mgy + =0 2m ∂x 2m ∂y ∂t The principle function is S(x. assuming the projectile is fired off at time t = 0 from the origin with the velocity v0 . p2 p2 y x + + mgy 2m 2m Following the examples in section 10. it is cyclic. α) − αt Expressed in terms of the characteristic function. Find both the equation of the trajectory and the dependence of the coordinates on time. α) = − 1 2 (2mα − αx − 2m2 gy)3/2 3m2 g 2 2mα − αx − 2m2 gy Thus our principle function is S(x. αx . S(x.2. First we find the Hamiltonian.17 Solve the problem of the motion of a point projectile in a vertical plane. t) = xαx + − Solving for the coordinates. t) = Wx (x. t) = xαx + Wy (y. α. making an angle θ with the horizontal. α. αx) + Wy (y. we have Wy (y. we get for our HamiltonianJacobi equation 2 1 ∂Wy 2 αx + ( ) + mgy = α 2m 2m ∂y This is ∂Wy = ∂y Integrated. using the Hamilton-Jacobi method. α) − αt Because x is not in the Hamiltonian. 1 2 (2mα − αx − 2m2 gy)3/2 − αt 3m2 g 4 . αx . x(t) = βx + x(0) = 0 → βx = − αx β m 2 α αx g − =0 y(0) = 0 → − β 2 + 2 mg 2mα g αx m y(0) = v0 sin θ = −gβ ˙ x(0) = v0 cos θ = ˙ Thus we have for our constants β= βx = α= v0 sin θ −g 2 v0 cos θ sin θ g 2 mg 2 2 mv0 2 (v0 sin θ + v0 cos2 θ) = 2g 2 αx = mv0 cos θ Now our y(t) is 2 g v0 sin θ 2 v0 v 2 cos2 θ y(t) = − (t +) + − 0 2 g g 2g 2 2 g v0 sin2 θ v0 v 2 cos2 θ g y(t) = − t2 + v0 sin θt − + − 0 2 2 g2 g 2g 5 .β= βx = ∂S 1 2 =− (2mα − αx − 2m2 gy)1/2 − t ∂α mg ∂S αx 2 = x + 2 (2mα − αx − 2m2 gy)1/2 ∂αx m g Solving for both x(t) and y(t) in terms of the constants β. βx . α and αx 2 g αx α y(t) = − (t + β)2 + − 2 mg 2m2 g x(t) = βx + Our x(t) is αx 1 2 (− (2mα − αx − 2m2 gy)1/2) m mg αx (β + t) m We can solve for our constants in terms of our initial velocity. and angle θ through initial conditions. φ. E. we get H= p2 p2 (pφ − pψ cos θ)2 ψ + M gh cos θ + θ + 2I3 2I1 2I1 sin2 θ Setting up the principle function. solved for the partial S's 2 αψ 1 ∂Wθ 2 (αφ − αψ cos θ)2 + ( ) + + M gh cos θ = E 2I3 2I1 ∂θ 2I1 sin2 θ Turning this inside out: 6 . and obtain the formal solution to the motion as given by Eq. αψ . (5.63). in the Hamilton-Jacobi mehtod. E) + ψαψ + φαφ − Et Using ∂S =p ∂q we have for our Hamilton-Jacobi equation. with one point fixed. u(t) t= u(0) du (1 − u2)(α − βu) − (b − au)2 Expressing the Hamiltonian in terms of momenta like we did in the previous problem. Answer: This is the form we are looking for. we see S(θ. t) = Wθ (θ.26 Set up the problem of the heavy symmetrical top. ψ.g y(t) = − t2 + v0 sin θt 2 and for x(t) x(t) = 2 v0 v0 sin θ cos θ sin θ + v0 cos θ + v0 cos θt g −g x(t) = v0 cos θt Together we have g y(t) = − t2 + v0 sin θt 2 x(t) = v0 cos θt 10. noting the cyclic coordinates. αφ . E) = ∂θ When integrated. the value of θ is θ(t) θ(t) t= θ(0) dθ (α − (b−a cos θ)2 sin2 θ − β cos θ)1/2 The integrand is the exact expression as Goldstein's (5.62).∂ Wθ (θ. Making the substitution u = cos θ we arrive home u(t) t= u(0) du (1 − u2)(α − βu) − (b − au)2 7 . 2I1 E − 2 αψ I1 I3 − (αφ − αψ cos θ)2 − 2I1 M gh cos θ sin2 θ Wθ = (2I1 E − 2 αψ I1 (αφ − αψ cos θ)2 − − 2I1 M gh cos θ)1/2 dθ I3 sin2 θ Now we are in a position to solve βθ = ∂Wθ = βθ + t = ∂E ∂Wθ ∂S = −t ∂E ∂E 2I1 dθ 2(2I1 E − α2 I1 ψ I3 − (αφ −αψ cos θ)2 sin2 θ − 2I1 M gh cos θ)1/2 Using the same constants Goldstein uses α= 2E − 2 αψ 2E − I1 I1 I3 I1 2M gl β= I1 α2 ψ I3 = where αφ = I1 b αψ = I1 a and making these substitutions βθ + t = (I1 (2E − βθ + t = I1 dθ α2 ψ I3) 2 − I1 (b−a cos θ) − I1 2M gh cos θ)1/2 sin2 θ 2 dθ (α − (b−a cos θ)2 sin2 θ − β cos θ)1/2 For time t. 27. which is Goldstein's (10. 2004 10. where F is a constant. 10. we have only the first quadrant. Using action-angle variables.Homework 12: # 10. Multiply this by 4 for all of phase space and our action variable J becomes E/F J =4 0 √ 2m E − F q dq A lovely u-substitution helps out nicely here. Solution: Define the Hamiltonian of the particle p2 + F |q| 2m Using the action variable definition. integrated from q = 0 to q = E/F (where p = 0).82): H≡E= J= we have J= 2m(E − F q) dq p dq For F is greater than zero.13 A particle moves in periodic motion in one dimension under the influence of a potential V (x) = F |x|. find the period of the motion as a function of the particle's energy. Cylinder Michael Good Nov 28. u = E − Fq 0 → √ 2mu1/2 du = −F dq 1 du −F √ 8 2m 3/2 du = E 3F J =4 E J= 4 2m F √ E u1/2 0 1 .13. .Goldstein's (10. τ= This is √ ∂ 8 2m 3/2 τ= [E] ∂E 3F And our period is √ 4 2mE τ= F 10. Express the motion in terms of J and its conjugate angle variable.95) may help us remember that ∂H =ν ∂J and because E = H and τ = 1/ν. involving two degrees of freedom. 2 Using the form of the Hamiltonian..(eq'n 10.. H= U (r0) = − l2 3 + V (r0) = 0 mr0 2 ∂J ∂E . U (r) the Hamiltonian becomes H= 1 2 p + U (r) 2m r The r0 from above will be some minimum of U (r). Taylor series go like 1 (x − a)2 f (a) + . f (x) = f (a) + (x − a)f (a) + 1 U (r) = U (r0) + (r − r0)U (r0) + (r − r0)2 U (r0) + .27 Describe the phenomenon of small radial oscillations about steady circular motion in a central force potential as a one-dimensional problem in the action-angle formalism.65) we have 1 2 l2 (p +) + V (r) 2m r r2 Defining a new equivalent potential. find the period of the small oscillations. 2! Lets expand around some r0 for our potential.. in polar coordinates. With a suitable Taylor series expansion of the potential. Solution: As a reminder. Thus our Hamiltonian becomes H= This is H= H= 1 1 2 p + U (r0) + (r − r0)2 U (r0) 2m r 2 1 2 p + U (r0 + λ) 2m r 1 2 1 p + U (r0) + λ2 U (r0) = E 2m r 2 If we use the small energy defined as = E − U (r0) We see 1 2 1 2 p + λ k 2m r 2 This energy is the effect on the frequency.The second derivative is the only contribution U = 3l2 4 + V (r0) = k mr0 where k > 0 because we are at a minimum that is concave up.6 = = We have for the action variable J = 2π and a period τ= with motion given by r = r0 + J sin 2πω πmω mJω cos 2πω π 3 ∂J = 2π ∂ m k m k Jω 2π pr = . so following section 10. If there is a small oscillation about circular motion we may let r = r0 + λ where λ will be very small compared to r0 . Solution: Trivially. one for θ and one for z. The time it takes to fall is the same time it takes to bounce up. T =2 2h g → νz = 1 2 g 2h To derive these frequencies via the action-angle formulation we first start by writing down the Hamiltonian for the particle. Eθ = 2 p2 Jθ θ = 2mR2 4π 2 2mR2 4 . Find the two frequencies of its motion using the action angle variable formulation. Breaking the energy into two parts. we know the frequency around the cylinder to be its angular speed divided by 2π because it goes 2π radians in one revolution. It is released and bounces around the perimeter. we may find the frequency of its up and down bouncing through Newtonian's equation of motion. H ≡ E = mgz + ˙ pθ = mθR2 we may write Jθ = 2πpθ based on Goldstein's (10.101). and because θ does not appear in the Hamiltonian.A particle is constrained to the edge of a cylinder. νθ = h= 1 2 gt 2 2h g t= Multiply this by 2 because the period will be measured from a point on the bottom of the cylinder to when it next hits the bottom of the cylinder again. by symmetry. and his very fine explanation. we may express the Eθ part as a function of Jθ . p2 p2 z θ + 2m 2mR2 Noting that pθ is constant because there is no external forces on the system. ˙ θ 2π And also simply. therefore it is cyclic and its conjugate momentum is constant. The frequency is νθ = ∂Eθ Jθ = 2 mR2 ∂Jθ 4π νθ = Thus we have ˙ Jθ 2πpθ pθ mθR2 = = = 4π 2 mR2 4π 2 mR2 2πmR2 2πmR2 ˙ θ 2π The second part is a bit more involved algebraically. its potential energy when released from rest. Only the second part remains: Jz = Solved in terms of Ez 3 Ez = (g 4 The frequency is νz = ∂Ez 2 3 =((g ∂Jz 3 4 m 2/3 1)) 1/3 2 J m Jz)2/3 2 4√ 1 3/2 2m E 3 mg z 5 . Expressing the energy for z: νθ = Ez = mgz + Solving for pz and plugging into J= we get Jz = 2. √ 2 −1 Jz = 2 2m (Ez − mgz)3/2 ( ) 3 mg h 0 p2 z 2m p dq √ 2m (Ez − mgz)1/2 dz we can do this closed integral by integrating from 0 to h and multiplying by The original energy given to it in the z direction will be mgh. Thus the first part of this evaluated integral is zero. As you may already see there are many different steps to take to simplify. I'll show one. Yay! Our two frequencies together νθ = νz = 1 2 ˙ θ 2π g 2h The condition for the same path to be retraced is that the ratio of the frequencies to be a rational number. that the m's cancel. 6 . and the constant part becomes g 1/2 h1/2 The number part simplifies down to 1 √ 2 2 Thus we have 1 νz = √ 2 2 1 g = h 2 g 2h as we were looking for from Newton's trivial method. Lets gather the numbers. 2 3 νz = ((g 3 4 m 2/3)) 2 [1 4 3g 2 3/2]1/3 m (mgh) Now we have a wonderful mess. with some careful observation. This is explained via closed Lissajous figures and two commensurate expressions at the bottom of page 462 in Goldstein.All we have to do now is plug what Jz is into this expression and simplify the algebra. and the constants to one side νz = 2 3 2/3 1 3(4) 21/3 4 (3)1/3 21/6 g 2/3 m1/3 g 1/3 m1/6 m1/2 g 1/2 h1/2 You may see.